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Applied Functional Analysis (150630)

Tuesday, January 27,2009 09.00-12.00

- An explanation to every answer is required. You can make use of a calculator.
- (Part of) the score to exercise 1(c) and 6(a),(b),(c) may be earned by homeworks; see the score table below (these are indicated by $\cdot$).

1. We consider the sequence (f_n) with $(f_n) \in C[-1,1]$ defined by

$$f_n(x) = (1 - |x|)^n$$

   (a) Does the sequence converge uniformly?

   (b) We equip $C[-1,1]$ with the norm $\|f\|_2 = \left(\int_{-1}^{1} f^2(x)dx\right)^{1/2}$. Does the sequence converge in $(C[-1,1], \| \, \|_2)$?

   (c) Show that $(C[-1,1], \| \, \|_2)$ is not complete

2. In the real Hilbert space $L^2([-1,1])$ the set

$$S = \left\{ g_0(x) = \frac{1}{\sqrt{2}}, g_n(x) = \cos(n\pi x), h_n(x) = \sin(n\pi x) \right\} n = 1, 2, 3, ...$$

forms a maximal orthonormal system. Consider the linear subspace E consisting of all even functions, so

$$E = \left\{ f \in L^2([-1,1]) | \; f(x) = f(-x) \quad \text{for almost all } x \in [-1,1] \right\}$$

   (a) Prove $\{g_0, g_n\}$ $n = 1, 2, 3, ...$is maximal orthonormal system in E

   (b) Define the linear operator $A : L^2[-1,1] \to L^2[-1,1]$ by $(Af)(x) = f(-x)$ for $f \in L^2[-1,1]$ and $x \in [-1,1]$. Show that A is bounded, find $\|A\|$

   (c) Prove that E is a closed linear subspace of $L^2([-1,1])$.
   (hint: view E as being the null space of an operator)

3. Suppose H is a real Hilbert space and $\varphi : H \to R$ is a non-zero bounded linear functional.

   (a) Prove that there is a $g \in (\mathcal{N}(\varphi))^{\perp}$ such that $\varphi(g) = 1$.

   (b) Let $f \in H$ with $\varphi(f) = 1$ and $f \neq g$. Prove that $\|f\| > \|g\|$

   (c) Find the real valued function $g \in L^2(0,1)$ such that $\int_{1/2}^{1} g(x)\,dx = 1$ and $\int_0^1 (g(x))^2\,dx$ minimal

4. Consider $A : L^2(0,1) \to L^2(0,1)$ given by

$$(Af)(x) = \int_0^1 e^{-ix-y} f(y)\,dy$$

   (a) Determine the adjoint A^* of A

   (b) Prove that $\|A\| < 1$

   (c) Explain that the integral equation $f(x) - \int_0^1 e^{-ix-y} f(y)\,dy = g(x)$ has a unique solution $f(x) \in L^2[0,1]$ for every choice $g(x) \in L^2[0,1]$

5. The linear operator $A : l^2 \to l^2$ is given by

$$A(x_1, x_2, x_3, x_4, ...) = \left(\frac{1}{2}x_2, \frac{1}{2}x_1, \frac{1}{4}x_4, \frac{1}{4}x_3, \frac{1}{6}x_6, \frac{1}{6}x_5, ...\right)$$

   so

$$(Ax)_{2n-1} = \frac{1}{2n}x_{2n} \quad \text{and} \quad (Ax)_{2n} = \frac{1}{2n}x_{2n-1}$$

   (a) Calculate A^*

   (b) Find the spectrum $\sigma(A)$ of A
   (hint: take first a look at the eigenvalues of A)

6. Let $(a_n)_{n\in\mathbf{N}}$ be a sequence of complex numbers.

   (a) Show that the operator $A : l^2 \to l^2$ with $A(x_1, x_2, ...) = (a_1x_1, a_2x_2, ...)$ is densely defined and closed

   (b) For what choices of (a_n) is A an unbounded operator?

   (c) Determine the domain of A^* and $A^*(y_1, y_2, ...)$

GRADING POINTS	1.(a)2	2.(a)2	3.(a)2	4.(a)2	5.(a)2	6.(a)2
	(b)2	(b)2	(b)2	(b)2	(b)3	(b)2
	(c)3	(c)2	(c)3	(c)1		(c)2

Total: 36+4=40 points.