Applied Functional Analysis (191506302)

Tuesday, January 25, 2011, 08.45-11.45

- An explanation to every answer is required
- You can make use of a calculator
- (Part of) the score to exercises 2 and 3(c) may be earned by homeworks; see the score table at the end

1. We define the space F of all 'finite sequences of real numbers' as follows:

$$F = \{\underline{x} = (x_1, x_2, ...) | x_n \in \mathbb{R}, \text{ there exist an } N \text{ such that } x_n = 0 \text{ for } n > N\}$$

View F as a linear subspace of the vector space l^1, but we define an alternative norm on F:

$$\|\underline{x}\| = \sum_{n=1}^{\infty} |nx_n|$$

(a) Check that $\| \ \|$ indeed defines a norm on F

(b) Show that $(F, \| \ \|)$ is not a Banach space

(c) Describe the completion E of $(F, \| \ \|)$ as a special subset of l^1 and prove your choice

2. The (complex) inner product space $C^1([0,1])$ consists of all differentiable functions $f : [0,1] \to \mathbb{C}$ with the natural vector space structure for functions and with

$$(f, g) = f(0)\overline{g(0)} + \int_0^1 f'(x)\overline{g'(x)}dx$$

(a) Check that $(\ , \ )$ defines a complex inner product

(b) The norm induced be $(\ , \ )$ is denoted as $\| \ \|$. Give an expression for (f, g) in terms of $\| \ \|$.

(c) Let $h(x) = x + ix^2$ for $x \in [0,1]$. Determine $\{h\}^\perp$

3. Consider the boundary-value problem for y:

$$\begin{cases} y' - y = f \\ y(0) = 0 \end{cases}$$

where f is a given function in $L^2(0,1)$.

(a) Show that $y(x) = \int_0^x e^{x-t} \cdot f(t)dt$ is a solution to the problem.
Define the integral operator $A : L^2(0,1) \to L^2(0,1)$ by $Ag(x) = \int_0^1 k(x,t)g(t)dt$ for $g \in L^2(0,1)$ where

$$k(x,t) = \begin{cases} e^{x-t} & \text{if } 0 \leq t < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq t \leq 1 \end{cases}$$

(b) Show that $y(x)$ from (a) equals $Af(x)$

(c) Prove that A is bounded

4. The linear operator on the real normed vector space l^2 is given by $A(\underline{b}) = (b_2, b_2, b_4, b_4, b_6, b_6, ...)$ if $\underline{b} = (b_1, b_2, b_3, ...)$.

   (a) Show that A is bounded and determine $\|A\|$

   (b) Calculate A^* and A^2

   (c) Is A a compact operator? Explain your answer

   (d) Determine all eigenvalues of A

5. Let $A : L^2[0,1] \to L^2[0,1]$ be the kernel operator $Af(x) = \int_0^1 k(x,y)f(y)dy$ with kernel

$$k(x,y) = \begin{cases} y(1-x) \text{ if } 0 \le y \le x \le 1 \\ x(1-y) \text{ if } 0 \le x \le y \le 1 \end{cases}$$

   It is known that the eigenvalues of A are the numbers $\lambda_n = \frac{1}{n^2\pi^2}$ with corresponding eigenfunctions

$$g_n(x) = \frac{1}{\sqrt{2}} \sin(n\pi x) \text{ for } n = 1, 2, 3, ...$$

   (a) Prove that for a given $g \in L^2[0,1]$ the solution of

$$\begin{cases} u'' = -g \\ u(0) = 0, u(1) = 0 \end{cases}$$

   is given by $u = Ag$

   (b) Show that for $\lambda \in \mathbb{C}, \lambda \neq 0$ and given $g \in L^2[0,1]$ we have

$$\begin{cases} u'' + \lambda u = g \\ u(0) = 0, u(1) = 0 \end{cases} \iff (A - \frac{1}{\lambda}\mathrm{Id})u = \frac{1}{\lambda}Ag$$

   (c) Determine for which $\lambda \in \mathbb{C}$ the problem

$$\begin{cases} u'' + \lambda u = g \\ u(0) = 0, u(1) = 0 \end{cases}$$

   has a unique solution for a given g, and describe this solution in terms of g and g_n.

GRADING POINTS

Total: 36+4=40 points

1.(a)2	2.(a)2	3.(a)3	4.(a)2	5.(a)3
(b)2	(b)2	(b)2	(b)2	(b)2
(c)2	(c)2	(c)3	(c)2	(c)3
			(d)2	