Applied Functional Analysis (191506302)

Tuesday, January 29, 2013, 08.45-11.45

- An explanation to every answer is required
- You can make use of a calculator
- (Part of) the score to exercises 2 and 6 may be earned by homeworks; see the score table at the end

1. We consider the space $C_b[0,\infty)$ of bounded continuous real valued functions on $[0,\infty)$ equipped with the norm $\|\cdot\|_\infty$ given by $\|f\|_\infty = \sup_{x\geq 0}|f(x)|$, for all $f \in C_b[0,\infty)$. Let E be the linear subspace of $C_b[0,\infty)$ consisting of all functions $f \in C_b[0,\infty)$ for which $\lim_{x\to\infty} f(x)$ exists.

   (a) Show that E is a closed subspace of $C_b[0,\infty)$.

   (b) The linear functional $\varphi : E \to \mathbb{R}$ is defined by $\varphi(f) = \lim_{x\to\infty} f(x)$. Show that φ is bounded and compute $\|\varphi\|$.

2. Let M be a linear subspace of a normed vector space E and N be a linear subspace of E^* (the dual of E). We define

$$M^0 = \{\varphi \in E^* | \varphi(x) = 0 \text{ for all } x \in M\}$$
$$N_0 = \{x \in E | \varphi(x) = 0 \text{ for all } \varphi \in N\}$$

   (a) Take $E = L^3(-1,1)$ and $M = \{f \in L^3(-1,1) | f(x) = 0 \text{ for all } 0 \leq x \leq 1\}$.
   Identify M^0 as a subspace of $L^{\frac{3}{2}}(-1,1)$, (the dual of $L^3(-1,1)$).
   What functions are element of M^0?

   (b) Show (in general) that $(M^0)_0 = M$ in case M is closed (hint: use the theorem of Hahn Banach).

3. Consider the real Hilbert space ℓ^2 with the usual inner product.

   (a) Let $T = \{y_1, y_2, ...\}$, where the vectors $y_1, y_2, ...$ in ℓ^2 are given by

$$y_1 = (1,0,-2,0,0,...), y_2 = (0,1,0,-2,0,0,...)$$
$$y_3 = (0,0,1,0,-2,0,0,...), \text{ etc.}$$

   Find an orthonormal basis (that is, a maximal orthonormal system) for $T^\perp$.

   (b) Let $z = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...)$. Find the vector $h \in T^\perp$ which satisfies

$$\|z - h\| = \min\left\{\|z - x\| : x \in T^\perp\right\}.$$

4. Consider the complex Hilbert space $L^2(-1,2)$ with the usual inner product given by $(f,g) = \int_{-1}^{2} f(t)\overline{g(t)}dt, f,g \in L^2(-1,2)$. The linear map $A : L^2(-1,2) \to L^2(-1,2)$ is defined by

$$(Af)(x) = (ix)f(x), \qquad f \in L^2(-1,2).$$

   (a) Show that A is bounded and compute $\|A\|$.

   (b) Determine the adjoint A^* of A.

5. Let $A : L^2(0,1) \to L^2(0,1)$ be the linear operator on the complex Hilbert space, defined by

$$Af(x) = \int_0^x f(t)dt \quad \text{for all } f \in L^2(0,1)$$

The adjoint of A is denoted by A^*.

   (a) Show that A does not have any eigenvalue.

   (b) Prove that A^*A is a self adjoint and compact operator.

   One can prove that the system $\{\varphi_k(x)|k = 0,1,2,...\}$ with $\varphi_k(x) = \sqrt{2}\cos((k+\frac{1}{2})\pi x)$ is a maximal orthonormal system in $L^2(0,1)$ of eigenfunctions of A^*A.

   (c) Give the spectral decomposition of A^*A.

6. By $\ell^2_{\mathbb{C}}$ we denote the normed complex vector space of all sequences $\underline{a} = (a_1, a_2, ...)$ of complex numbers with $\|\underline{a}\| = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}$ is finite. Define $A : \ell^2_{\mathbb{C}} \to \ell^2_{\mathbb{C}}$ by $A\underline{a} = (a_1, 2a_2, 3a_3, ...)$. Determine whether A is a closed operator where the domain $\mathcal{D}(A) = \{\underline{a} \in \ell^2_{\mathbb{C}} | \sum_{n=1}^{\infty} |na_n|^2 \text{ is finite}\}$.

7. Let E be a (complex) Banach space and $A \in BL(E,E)$.

   (a) Suppose λ_1 and λ_2 are different eigenvalues of A with corresponding eigenvectors f_1 and f_2, respectively. Show that f_1 and f_2 are linearly independent.

   (b) Suppose A is invertible. Show that

$$\sigma(A^{-1}) = \left\{\frac{1}{\lambda} | \lambda \in \sigma(A)\right\}$$

   (here $\sigma(A)$ is the spectrum of A).

GRADING POINTS

Total: 36+4=40 points

1.(a)3	2.(a)3	3.(a)3	4.(a)3	5.(a)2	6.3	7.(a)1
(b)3	(b)3	(b)2	(b)3	(b)2		(b)3
				(c)2		