Exam: Continuous Optimisation 2015

1. Let $f : \mathcal{C} \to \mathbb{R}$, $\mathcal{C} \subset \mathbb{R}^n$ convex, be a convex function. Show that then the following holds: [3 points]
   A local minimizer of f on $\mathcal{C}$ is a global minimizer on $\mathcal{C}$. And a strict local minimizer of f on $\mathcal{C}$ is a strict global minimizer on $\mathcal{C}$.

2. (a) Show that for $\mathbf{d} \in \mathbb{R}^n$ it holds: [2 points]

$$\mathbf{d}^\mathsf{T}\mathbf{x} \geq 0 \;\forall \mathbf{x} \in \mathbb{R}^n \quad \Leftrightarrow \quad \mathbf{d} = 0.$$

   (b) Let $\mathbf{c}, \mathbf{a}_i \in \mathbb{R}^n, i = 1, \ldots, m$ $(m \geq 1)$. Show using the Farkas Lemma (lecture sheets, Th. 5.1) that precisely one of the following alternatives (I) or (II) is true: [3 points]
   (I): $\mathbf{c}^\mathsf{T}\mathbf{x} < 0,\ \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq 0, i = 1, \ldots, m$ has a solution $\mathbf{x} \in \mathbb{R}^n$.
   (II): there exist $\mu_1 \geq 0, \ldots, \mu_m \geq 0$ such that: $\mathbf{c} + \sum_{i=1}^m \mu_i \mathbf{a}_i = 0$

3. Given is the problem

$$(P) \qquad \min_{\mathbf{x}\in\mathbb{R}^2} \ (-2x_1 - x_2) \qquad \text{s.t.} \qquad x_1 \leq 0, \ \text{and} \ -(x_1-1)^2 - (x_2-1)^2 + 2 \leq 0 \ .$$

   (a) Is (P) a convex problem? Sketch the feasible set and the level set of f given by $f(\mathbf{x}) = f(\overline{\mathbf{x}})$ with $\overline{\mathbf{x}} = 0$. Is LICQ (constraint qualification) satisfied at $\overline{\mathbf{x}}$? [3 points]
   (b) Show that the point $\overline{\mathbf{x}} = 0$ is a KKT-point of (P). Determine the corresponding Lagrangean multipliers. [3 points]
   (c) Show that $\overline{\mathbf{x}}$ is a local minimizer. What is the order of this minimizer? Is it a global minimizer? [3 points]
   (d) Consider now the program (objective f and constraint function g_2 interchanged): [2 points]

$$(\widetilde{P}) \qquad \min_{\mathbf{x}\in\mathbb{R}^2} -(x_1-1)^2 - (x_2-1)^2 + 2 \qquad \text{s.t.} \qquad x_1 \leq 0, \ \text{and} \ -2x_1 - x_2 \leq 0 \ .$$

   Explain (without any further calculations) why $\overline{\mathbf{x}} = 0$ is also a local minimizer of $(\widetilde{P})$.

4. Consider the (nonlinear) program: [3 points]

$$(P) \qquad \min_{\mathbf{x}} \ f(\mathbf{x}) \qquad \text{s.t.} \quad \mathbf{x} \in \ \mathcal{F} := \{\mathbf{x} \in \mathbb{R}^n \mid g_j(\mathbf{x}) \leq 0, \ j \in J\}$$

   with $f, g_j \in C^1$, $f, g : \mathbb{R}^n \to \mathbb{R}$, $J = \{1, \ldots, m\}$. Let $\mathbf{d}_k$ be a strictly feasible descent direction for $\mathbf{x}_k \in \mathcal{F}$. Show that for $t > 0$, small enough, it holds:

$$f(\mathbf{x}_k + t\mathbf{d}_k) < f(\mathbf{x}_k) \quad \text{and} \quad \mathbf{x}_k + t\mathbf{d}_k \in \mathcal{F}$$

5. For a given nonempty set $\mathcal{A} \subseteq \mathbb{R}^n$ we define its conic hull, conic($\mathcal{A}$) by

$$\text{conic}(\mathcal{A}) := \left\{ \sum_{i=1}^{m} \mu^i \mathbf{x}^i \ : \ \mathbf{x}^i \in \mathcal{A},\ \mu^i \geq 0 \text{ for all } i,\ m \in \mathbb{N} \right\}.$$

   (a) Show that conic($\mathcal{A}$) is a convex cone. [2 points]

   (b) Show that if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathbb{R}^n$, with $\mathcal{B}$ being a convex cone, then conic($\mathcal{A}$) $\subseteq \mathcal{B}$. [3 points]

   (c) Show that conic($\mathcal{A}$) is full dimensional if and only if there does not exist $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{y}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathcal{A}$. [1 point]

6. In this question we will consider the proper cone $\mathcal{K} \subseteq \mathbb{R}^{n+2}$ defined as

$$\mathcal{K} = \left\{ \begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix} \ : \ \mathbf{y} \in \mathbb{R}^n,\ x, z \in \mathbb{R},\ \|\mathbf{y}\|_2 \leq x,\ z \geq 0 \right\}.$$

   (a) Consider a ray $\mathcal{R} = \{\mathbf{c} - y_1 \mathbf{a} \mid y_1 \in \mathbb{R}_+\}$ with fixed $\mathbf{a}, \mathbf{c} \in \mathbb{R}^n$. We wish to find the distance between the origin and the closest point in this ray. Formulate this problem as a conic optimisation problem over $\mathcal{K}$. [2 points]

   (b) Give an explicit characterisation of $\mathcal{K}^*$. [1 point]
   [Justification for your answer must be provided]

   (c) What is the dual problem to your formulation in part (a)? [2 points]
   [If you were not able to answer parts (a) and (b) then instead find the dual to: $\min_y \quad y \quad \text{s.t.} \quad \mathbf{c} + y\mathbf{a} \in \mathbb{R}^n_+$.]

7. Consider the following optimisation problem: [3 points]

$$\begin{aligned} \min_{\mathbf{x}} \quad & 2x_2^2 + 5x_1x_2 - 4x_2 \\ \text{s.t.} \quad & 2x_1^2 + x_1 + 3x_2^2 - 2x_1x_2 = 3 \\ & \mathbf{x} \in \mathbb{R}^2. \end{aligned} \tag{A}$$

   Give the standard positive semidefinite approximation for this problem, the solution of which would provide a lower bound to the optimal value of problem (A).

8. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	8	Total
Points:	3	5	11	3	6	5	3	4	40

A copy of the lecture-sheets may be used during the examination.
Good luck!