Exam: Continuous Optimisation 2016

Monday 12$^{\text{th}}$ December 2016

1. We will consider the first step in iterative methods from $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to attempt to minimise the function $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(\mathbf{x}) = 2x_1^2 + x_2^2 \exp(x_1) - x_1 - x_2 \quad$ over $\mathbb{R}^2$.
   (a) Starting from $\mathbf{x}_0$, considering the direction of steepest descent, $\mathbf{d}_S$, as the search direction and exact line search (i.e. $\lambda_0 \in \arg\min_{\lambda\in\mathbb{R}}\{f(\mathbf{x}_0 + \lambda\mathbf{d}_S)\}$), evaluate $\mathbf{x}_1 = \mathbf{x}_0 + \lambda_0\mathbf{d}_S$. [2 points]
   (b) Starting from $\mathbf{x}_0$, considering Newton's direction, $\mathbf{d}_N$, as the search direction (not normalised), and $\lambda_0 = 1$, evaluate $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{d}_N$. [2 points]

2. (a) Consider two convex sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}$, and two convex functions $h : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathbb{R}$, with g also being a monotonically increasing function on $\mathcal{B}$. [3 points]
   For $f : \mathcal{A} \to \mathbb{R}$ given by $f(\mathbf{x}) = g(h(\mathbf{x}))$, show that f is a convex function.
   (b) For a norm $\| \bullet \|$ on $\mathbb{R}^n$ and a convex function $f : \mathbb{R}^n \to \mathbb{R}$, consider using the barrier method to solve the problem $\min_{\mathbf{x}}\{f(\mathbf{x}) : \|\mathbf{x}\| \leq 1\}$.
   Let $\widehat{\mathcal{F}} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < 1\}$ and $b : \widehat{F} \to \mathbb{R}$ be given by $b(\mathbf{x}) = (1 - \|\mathbf{x}\|)^{-2}$.
      i. Justify that b is a valid barrier function for this problem. [1 point]
      ii. Show that b is a convex function. [2 points]

3. Consider the problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & x_2 \\ \text{s.t.} \quad & x_1^2 \leq x_1 + x_2 \\ & 2x_1 \leq x_1^2 + x_2 \end{aligned} \tag{P}$$

   (a) Is (P) a convex optimisation problem? Justify your answer. [2 points]
   (b) Find a strictly feasible descent direction for the problem (P) at $\widehat{\mathbf{x}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. [2 points]
   (c) i. Show that the Linear Independency Constraint Qualification holds at all feasible points of (P). [2 points]
      ii. Find the KKT points for (P). [3 points]
      iii. Given that the optimal solution to (P) is attained, find the global minimiser and optimal value to this problem. Justify your answer. [1 point]
      iv. Provide justification for this global minimiser being a strict local minimiser of order 1. [1 point]
   (d) Formulate and solve the Lagrangian dual problem to (P). Is there strong duality? [4 points]

4. For $n \in \mathbb{N}$, consider a proper cone $\mathcal{L} \subseteq \mathbb{R}^n$ and a nonsingular matrix $\mathsf{A} \in \mathbb{R}^{n\times n}$. We then let $\mathcal{K} = \mathsf{A}\mathcal{L} := \{\mathsf{A}\mathbf{x} : \mathbf{x} \in \mathcal{L}\} \subseteq \mathbb{R}^n$.
   (a) Show that $\mathcal{K}$ is a convex cone. [1 point]
   (b) Show that $\mathcal{K}$ is pointed. [1 point]
   (c) Find $\mathcal{K}^*$, the dual cone to $\mathcal{K}$, in terms of $\mathcal{L}^*$. [2 points]
   (d) Show that $\mathcal{K}^*$ is pointed. [2 points]
   (e) Show that $\mathcal{K}$ is a proper cone. (You may assume that $\mathcal{K}$ is closed.) [1 point]

5. For $\mathbf{b} \in \mathbb{R}^m$ and $\mathsf{A}_1, \dots, \mathsf{A}_m \in \mathcal{S}^n$, consider the problem of varying $\mathbf{y} \in \mathbb{R}^m$ in order to minimise $\mathbf{b}^\mathsf{T}\mathbf{y}$, with the constraint that all the eigenvalues of $\sum_{i=1}^m y_i \mathsf{A}_i$ are between minus one and plus two.
   (a) Formulate this problem as a conic optimisation problem in a standard form. [2 points]
   (b) Find the dual problem to this conic optimisation problem. [2 points]
   If you were unable to solve part (a), then as an alternative question to (b): Find the dual problem to $\max_{\mathbf{y}}\{\mathbf{b}^\mathsf{T}\mathbf{y} : (\ \mathbf{c},\ \mathsf{C}\ ) + \sum_{i=1}^m y_i\ (\ \mathbf{a}_i,\ \mathsf{A}_i\ ) \in \mathbb{R}^p_+ \times \mathcal{PSD}^n\}$, with the vectors $\mathbf{c}, \mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}^p$ and the matrix $\mathsf{C} \in \mathcal{S}^n$.

6. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	Total
Points:	4	6	15	7	4	4	40

A copy of the lecture-sheets may be used during the examination. You may use any results from the lecture slides in your answers (Lemmas, Theorems, Corollaries, Exercises, etc.), however you should reference the result.

Good Luck!

Hints:

1. *g is a monotonically increasing function on $\mathcal{B} \subseteq \mathbb{R}$ if for all $a, b \in \mathcal{B}$ with $a \leq b$ we have $g(a) \leq g(b)$.*
2. *If g is differentiable in $\mathcal{B} \subseteq \mathbb{R}$ then g is a monotonically increasing function on $\mathcal{B}$ if and only if $g'(z) \geq 0$ for all $z \in \mathcal{B}$.*
3. *One of the properties of a norm is that it is a continuous function.*
4. $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \dfrac{1}{ac - b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$
5. *The following are equivalent for $\mathsf{A} \in \mathbb{R}^{n\times n}$:*
   - A *is a nonsingular matrix;*
   - A *has an inverse matrix;*
   - A^T *has an inverse matrix.*