First Name - (PRINT)	Surname - (PRINT)	University (affiliation)	Student ID

Mastermath
Continuous Optimization, Fall 2013
Final Exam

Instructor: Juan Vera
Date: Monday, 20 January, 2014
15:15 - 18:15

Instructions:

1. DURATION = 180 min.
2. This exam has 6 pages including this one
3. Clearly fill up the boxes above with your information
4. All answers should be written in this examination booklet in the indicated spaces. Use the extra blank pages if you need extra space
5. Please state all your assumptions and show all your work. Make sure to indicate this. Partial credit will be given if the reasoning is correct, even if the solution is not; however, none will be given for unsubstantiated claims, even if perfectly correct.
6. Notes (and/or books) are not allowed.
7. Electronic devices are not allowed.
8. Make sure that your hand writing is neat and readable, and your drawings (if any) are clear.
9. The problems were ordered at random.

Do not write in this space:

Problem	Points	Maximum
1		24
2		18
3		14
4		24
Total		80

1. **(24 marks)** Let $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. Consider the problem

$$\textbf{(P)} \quad \begin{array}{ll} \min_x & c^{\mathrm{T}} x \\ \text{s.t.} & Ax = b \\ & \|x\|^2 \leq 1 \end{array}$$

Let $L(x, y, z) = c^{\mathrm{T}} x + y^{\mathrm{T}}(Ax - b) + z(\|x\|^2 - 1)$ be the Lagrangian of (P).

(a) **[4 points]** Is (P) convex?

(b) **[4 points]** Is L convex?

(c) **[4 points]** Write the KKT conditions for (P).

(d) **[6 points]** Compute $g(y, z)$ the dual function of (P).

(e) **[2 points]** Write the dual problem for (P).

(f) **[4 points]** Does strong duality hold?

2. **(18 marks)**

   (a) **[10 points]** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function.

      i. Let $x, s \in \mathbb{R}^n$. Define $\phi(t) = f(x + ts)$. Show that $\phi'(t)$ is monotonically non-decreasing for all t.

      ii. For any $t > 0$ and $x \in R^n$, define $g(x, t) = tf(x/t)$. (Notice that $\text{Dom}\, g = \mathbb{R}^n \times \mathbb{R}_{++}$.) Show that g is convex.

   (b) **[8 points]** Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be strictly convex functions. Let $t_1 \in \mathbb{R}$ be the minimizer of f_1 and t_2 be the minimizer of f_2. Assume $t_1 \leq t_2$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) = f_1(t) + f_2(t)$. Are the following statements true? (remember to justify your answers).

      i. f is strictly convex

      ii. Let t be a minimizer of f. Then $t_1 \leq t \leq t_2$.

3. **(14 marks)**

(a) **[8 points]** Consider the convex optimization problem

$$\textbf{(CO)} \quad \begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & g_i(x) \leq 0 \end{array}$$

Let $L(x, y)$ be the Lagrangian of (CO). Let $(\hat{x}, \hat{y})$ be a KKT solution for (CO). Show that for any (CO)-feasible point,

$$L(\hat{x}, \hat{y}) \leq L(x, \hat{y}) \leq f(x).$$

(b) **[6 points]** Consider the following pair of equivalent problems

$$\textbf{(P-orig)} \quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x) \leq b \end{array} \qquad \textbf{(P-cube)} \quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x)^3 \leq b^3 \end{array}$$

i. Assume (x^*, y^*) is KKT point for (P-cube). Find a KKT point for (P-orig).

4. **(24 marks)** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous differentiable function. Assume that for some α the sub-level set $D_\alpha = \{x : f(x) \leq \alpha\}$ is bounded. Let H be some positive semidefinite $n \times n$ matrix and I be the $n \times n$ identity matrix. Consider the following algorithm to find $\min_x f(x)$.

```
Require: x^0 ∈ D_α: initial point. H ⪰ 0.
  k ← 0.
  while some non-optimality condition do
    Let s^k = −((1/k)H + I)∇f(x^k).
    Do line search: Let λ^k = arg min_{λ≥0} f(x^k + λs^k).
    Let x^{k+1} = x^k + λ^k s^k
    k ← k + 1
  end while
  return x^k
```

Assume that the non-optimality condition used as stoping criterium is $\nabla f(x_k) \neq 0$.

(a) **[10 points]** Show that in each iteration:

i. $(s^k)^{\mathrm{T}} \nabla f(x^k) < 0$.

ii. λ^k exists and $\lambda_k > 0$.

iii. $(s^k)^{\mathrm{T}} \nabla f(x^{k+1}) = 0$.

(b) **[6 points]** Assume $\bar{x} = \lim_{k \to \infty} x^k$ and $\bar{s} = \lim_{k \to \infty} s^k$ exist. Show that $\nabla f(\bar{x}) = 0$.

(c) **[2 points]** Show that if f is convex then $\bar{x}$ is a global minimizer of f.

(d) **[6 points]** Explain why the optimality condition used is not practical. Give a practical optimality condition that will ensure "almost optimality". Explain in what sense your solution is almost optimal.