Kenmerk : TW05/MHV/ConTiFi
Datum : October 24, 2006

# Answers Exam Continuous Time Finance

Code : 151530
Date : 9 november 2005

1. Let $C(t,S)$ denote the Black-Scholes call option formula and $\Delta(t,S)$ the corresponding delta. We know that the call-option can be replicated using stock and bond by investing $\Delta(t,S_t)$ in stocks, i.e. a percentage $u_t^S = S_t\Delta(t,S_t)/C(t,S_t)$ so

$$\frac{dC(t,S_t)}{C(t,S_t)} = u_t^S\frac{dS_t}{S_t} + (1-u_t^S)\frac{dB_t}{B_t}$$

Rewriting gives

$$\frac{dS_t}{S_t} = \frac{C(t,S_t)}{S_t\Delta(t,S_t)}\frac{dC(t,S_t)}{C(t,S_t)} + (1-\frac{C(t,S_t)}{S_t\Delta(t,S_t)})\frac{dB_t}{B_t}$$

so the self-financing strategy is given by

$$\begin{aligned} dS_t &= \frac{1}{\Delta(t,S_t)}dC(t,S_t) + \frac{S_t - C(t,S_t)/\Delta(t,S_t)}{B_t}dB_t \\ S_t &= \frac{1}{\Delta(t,S_t)}C(t,S_t) + \frac{S_t - C(t,S_t)/\Delta(t,S_t)}{B_t}B_t \end{aligned}$$

2. a. Use the fact that $\{\omega : M_t(\omega)\le a\} = \{\omega : M_t(\omega)\le a \& W_t(\omega)\le a\}$.

   b. The pricd $P$ equals

$$\begin{aligned} P &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{\max_{u\in[0,T]} S_u\ge L\}} \mid \mathcal{F}_0] = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{\max_{u\in[0,T]} W_u\ge \ln(L/S_0)/\sigma\}}] \\ &= e^{-rT}\mathbb{Q}[\max_{u\in[0,T]} W_u \ge \ln(L/S_0)/\sigma] = 2e^{-rT}N(\frac{\ln(S_0/L)}{\sigma\sqrt{T}}) \end{aligned}$$

   c. When $\sigma\to\infty$ we have that $P$ goes to $e^{-rT}$ since infinite volatility means that the probability that the level $L$ will be hit goes to one, so we receive a sure payoff of 1 at $T$ in the limit.

   d. Take 1000 times the $\Delta$ i.e. $1000\frac{\partial P}{\partial S_0}$ in the expression above.

3. a. See Björk for Martingale Representation Theorem.

   b. If $V$ is tradeable then there exist $h^S$ and $h^B$ processes such that

$$\begin{aligned} dV &= h^S dS + h^B dB \\ V &= h^S S + h^B B \end{aligned}$$

   Since $S/B$ is a $\mathbb{Q}$-martingale, there exists a process $\zeta$ such that $d\frac{S}{B} = \zeta dW$ where $W$ is Brownian Motion under $\mathbb{Q}$. but then using Ito,

$$\begin{aligned} d\frac{V}{B} &= \frac{h^S}{B}dS + (\frac{h^B}{B} - \frac{V}{B^2})dB = \frac{h^S}{B}dS - \frac{h^S S}{B^2}dB \\ d\frac{S}{B} &= \frac{1}{B}dS - \frac{S}{B^2}dB \end{aligned}$$

   so $d\frac{V}{B} = h^S d\frac{S}{B} = h^S\zeta dW$. which shows that $V/B$ is a martingale under $\mathbb{Q}$.

   c. Ito shows that $d\frac{M}{B} = \frac{\phi}{B}dm$ so since $M/B$ is a martingale under $\mathbb{Q}$, the same must be true for $m$ by Martingale Representation.

   d. We have by c.

$$m_t = \mathbb{E}^{\mathbb{Q}}[M_T \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[G_T \mid \mathcal{F}_t]$$

e. The price is

$$
\begin{aligned}
m_0 &= \mathbb{E}^{\mathbb{Q}}[C(t_1, S_{t_1}) \mid \mathcal{F}_0] \;=\; \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}[e^{-r(T-t_1)}(S_T - K)^+ \mid \mathcal{F}_{t_1}] \mid \mathcal{F}_0\right] \\
&= e^{rt_1}\mathbb{E}^{\mathbb{Q}}[e^{-rT}(S_T - K)^+ \mid \mathcal{F}_0] \;=\; e^{rt_1}C(0, S_0)
\end{aligned}
$$

where $C(t, S)$ is the Black-Scholes call formula for maturity $T$ and strike $K$.

4. a. The description of the model implies that

$$S_t - De^{-r(t_D - t)}\mathbf{1}_{\{t \le t_D\}} \;=\; ae^{\sigma W_t + (r - \frac{1}{2}\sigma^2)t}$$

but inserting $t = 0$ gives

$$S_0 - De^{-rt_D} \;=\; a$$

and substituting this gives

$$S_t \;=\; (S_0 - De^{-rt_D})e^{\sigma W_t + (r - \frac{1}{2}\sigma^2)t} + De^{-r(t_D - t)}\mathbf{1}_{\{t \le t_D\}}$$

b. The price is

$$P \;=\; e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ \mid \mathcal{F}_0]$$

but sibstiuting the expression above then gives

$$P \;=\; e^{-rT}\mathbb{E}^{\mathbb{Q}}[((S_0 - De^{-rt_D})e^{\sigma W_T + (r - \frac{1}{2}\sigma^2)T} - K)^+ \mid \mathcal{F}_0]$$

but we recognize this: if $C(t, S)$ is the ordinary Black-Scholes call formula for maturity $T$ and strike $K$ then this equals $C(0, S_0 - De^{-rt_D})$.

**Grading:**

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**Total:** 40 points