Kenmerk : TW05/MHV/ConTiFi
Datum : November 5, 2005

Exam Continuous Time Finance

Code : 151530
Date : 9 november 2005

All answers must be motivated.
Lots of success !

1. Consider a standard Black-Scholes model where

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma dW_t \\ \frac{dB_t}{B_t} &= r dt \end{aligned}$$

with $\{W_t,\, t \geq 0\}$ a standard Brownian Motion process under $\mathbb{P}$, and $\mu > r$ and $\sigma > 0$ known constants. Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by the Brownian Motion, and denote the equivalent martingale measure for this market by $\mathbb{Q}$. Assume that you want to replicate one stock S using call options (which all have the same maturity T and strike K) and the bankacount B. Give the correct self-financing replication strategy.

2. Remember that a standard Brownian Motion process W_t and its corresponding running maximum

$$M_t = \max_{u\in[0,t]} W_u$$

have the joint distribution function

$$\mathbb{P}(W_t \leq a,\, M_t \leq b) \quad = \quad N(\tfrac{a}{\sqrt{t}}) - N(\tfrac{a-2b}{\sqrt{t}}) \qquad (1)$$

but **this is valid only for values where** $b \geq 0$ **and** $a \leq b$. Consider the standard Black-Scholes model of the previous question. We want to determine the price, at time $t = 0$, of a contingent claims which pays at the time of maturity T

- 1, if at **any** time $u \in [0, T]$ the stock price S_u is larger than or equal to L, and pays
- zero otherwise.

You may assume that $0 < S_0 < L$, that $T > 0$ and that

$$r \quad = \quad \tfrac{1}{2}\sigma^2$$

which should simplify your calculations considerably.

a. Prove *in detail* why (1) implies that for all $a \geq 0$

$$\mathbb{P}(M_t \leq a) \quad = \quad 2N(\tfrac{a}{\sqrt{t}}) - 1$$

b. Determine an explicit formula for the price of the contingent claim given above in terms of the given parameters.

c. Explain what the correct limit should be for the price in b. when $\sigma \to \infty$ and when $S_0 = L$, and use this to check your formula.

d. Find an expression for the amount of stocks you should buy today (i.e. at time zero) if you want to hedge 1000 contracts of this claim.

3. Remember that we say that m is the futures price process for delivery of an asset G at time T if and only if

- $m_T = G_T$ and
- for all adapted left-continuous trading stratgies ϕ_t the following process M is a tradeable:

$$\begin{aligned} dM_t &= M_t \frac{dB_t}{B_t} + \phi_t dm_t \\ M_0 &= 0 \end{aligned}$$

We now take the Black-Scholes market of the first question. By definition, the martingale measure $\mathbb{Q}$ makes the discounted asset price process S_t/B_t a martingale. Remember that we call an asset a tradeable in this market if it can be replicated self-financingly using S and B.

a. State the Martingale Representation Theorem.

b. Prove: if an asset V is a tradeable, then its discounted asset price process V_t/B_t is a martingale under $\mathbb{Q}$.

c. Use this to prove that m is a martingale under $\mathbb{Q}$.

d. Prove that

$$m_t = \mathbb{E}^{\mathbb{Q}}[G_T \mid \mathcal{F}_t]$$

e. Let C be a call option on S with maturity T and strike K. Let F_0 be the futures price today ($t = 0$) for delivery of the call C at a time $t_1 \in]0, T[$. Derive an explicit formula for F_0.

4. Take again the Black-Scholes model formulated in the first question. We want to price a call option C on the asset S (with strike K and maturity T) but we now assume that the asset S will pay a single cash dividend with a value of D at time $t_D \in]0, T[$. The Escrowed Dividend Model assumes that *the asset price minus the present value of all dividends to be paid until the maturity of the option* follows a Geometric Brownian Motion under $\mathbb{Q}$ of the form

$$a e^{\sigma W_t + (r - \frac{1}{2}\sigma^2)t}$$

where $a > 0$ is a constant.

a. Show that this implies that

$$S_t = (S_0 - De^{-rt_D}) e^{\sigma W_t + (r - \frac{1}{2}\sigma^2)t} + De^{-r(t_D - t)} \mathbf{1}_{\{t \le t_D\}}$$

b. Let $F(S, t)$ be the standard Black-Scholes European call option formula for strike K and time to maturity T if the current time is t and the current stock price is S. Give a formula for the price at $t = 0$ of the call C defined above, in terms of F.

Grading:

1		:	4	2	a	:	2	3	a	:	3	free		:	4
					b	:	4		b	:	4				
4	a	:	3		c	:	2		c	:	3				
	b	:	2		d	:	2		d	:	3				
									e	:	4				

Total: 40 points