Exam Game Theory (191521800)
January 21, 2021; University of Twente

Motivate all your answers.

This exam has 9 exercises.
Motivate all your answers! You may not use any electronic device.
This exam comes with a cheat sheet that contains most of the basic definitions. (See the last pages.) Other necessary definitions are given in the questions. You are also allowed to bring your own cheat sheet (1 A4).

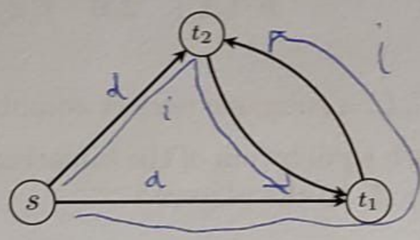
Noncooperative Game Theory

1. (5 points) Consider the (symmetric) bimatrix game given by

$$(A,B) = \begin{pmatrix} -1,1 & 14,9 \\ 7,16 & 12,14 \end{pmatrix}$$

   (a) Compute all Nash equilibria of this game.
   (b) Write down all conditions that define the correlated equilibria of this game.
   (c) Give a correlated equilibrium that is not a Nash equilibrium.

2. (4 points) Consider the network routing game depicted below. There are two players $i = 1, 2$, each of which has to select one of the two (s, t_i) paths. Let us call the available actions d for direct, and i for indirect. All arcs e in the network have a cost $c_e(x) = 3x$, with x as the number of players choosing arc e.



   Suppose that player 1 chooses first, and then player 2. Model this game as an extensive form game, set up the payoff matrix of the corresponding strategic form game, and compute the pure Nash and subgame perfect equilibria.

3. (6 points) Consider an arbitrary 2-player noncooperative game (A, B) with $A, B \in \mathbb{Z}^{m\times n}$, that is, all payoffs are integer. Prove or give a counterexample:

   (a) Is it true that such a game always has a pure strategy Nash equilibrium?
   (b) Is it true that such a game always has a Nash equilibrium $(\mathbf{p}, \mathbf{q})$ with $\mathbf{p} \in \mathbb{Q}^m$ and $\mathbf{p} \in \mathbb{Q}^n$?

Cooperative Game Theory

4. (4 points) Consider the following three player cooperative game (N, v).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	2	5	14	15	11	20

(a) Is the game essential? Is it superadditive? Is it convex?

(b) Compute the core $C(N, v)$, and expres it as convex hull of its extreme points. What is the domination core $DC(N, v)$ for this game?

5. (3 points) Compute the Shapley value ϕ for the three player cooperative game of the previous question. Is $\phi \in C$? Is the weber set $W = C$?

6. (4 points) Consider the following solution value ψ defined for a cooperative game (N, v) with $|N| = n$. For all $i \in N$, let i's payoff be

$$\psi_i(N, v) := \frac{v(\{i\})}{\sum_{j=1}^{n} v(\{j\})} v(N) .$$

Show that ψ is efficient and symmetric and fulfills the null player property, but may fail to be additive (in general).

Stochastic Game Theory

7. (3 points) Consider the repeated game $G^{\infty}(\delta)$ with

$$G = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} 2,3 & 0,2 & 0,5 \\ 3,1 & 2,0 & 1,2 \end{pmatrix} \end{array}$$

For which values of δ is (T, L) a subgame perfect equilibrium of $G^{\infty}(\delta)$?

(Notice that the unique Nash equilibrium of the bimatrix game G is (B, R). The action-pair (T, L) gives better payoffs to the players.)

8. (4 points) Consider the following zero-sum stochastic game with an infinite horizon and the discounted-reward criterion with discount factor $\beta = 0.9$.

2 $(\frac{2}{3}, \frac{1}{3})$	6 $(\frac{1}{3}, \frac{2}{3})$
3 $(\frac{1}{3}, \frac{2}{3})$	1 $(\frac{2}{3}, \frac{1}{3})$

state 1

$-\frac{1}{3}$ $(0, 1)$

state 2

(a) Determine the value of this game and optimal strategies for the players.

(b) Let $\mathbf{f} = ((1,0),(1))$ and $\mathbf{g} = ((0,1),(1))$ be decision rules for players 1 and 2, respectively. Write down the equations that you would have to solve to determine $v_\beta(\mathbf{f}, \mathbf{g})$. (You need not solve these equations.)

9. (3 points)

(a) Mention one (nontrivial) difference and one similarity between repeated games and infinite-horizon zero-sum stochastic games with the discounted-reward criterion.

(b) Consider an irreducible stochastic game with the average-reward criterion. Let $(\mathbf{f}, \mathbf{g})$ be a pair of stationary strategies. Show that if $v\mathbf{1}_N = v_\alpha(\mathbf{f}, \mathbf{g})$ and $\mathbf{w} = (I - P(\mathbf{f}, \mathbf{g}) + Q(\mathbf{f}, \mathbf{g}))^{-1}(\mathbf{r}(\mathbf{f}, \mathbf{g}) - v\mathbf{1}_N)$ then $\mathbf{w} + v\mathbf{1}_N = \mathbf{r}(\mathbf{f}, \mathbf{g}) + P(\mathbf{f}, \mathbf{g})\mathbf{w}$.