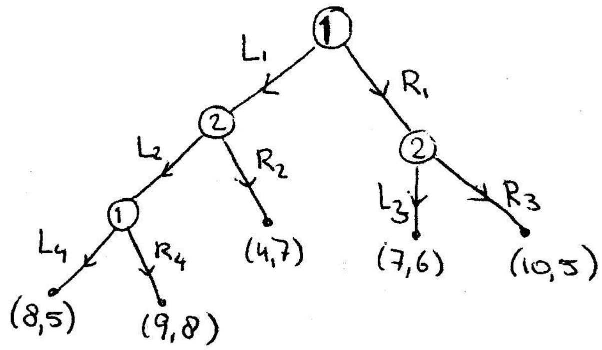
Course : **Game Theory**
Code : 191521800
Date : January 19, 2011
Time : 08.45-11.45 hrs

This exam consists of 6 exercises. Motivate all your answers.

1. Consider the extensive form game with two players in the figure below. (In this figure, the numbers in the circles indicate which player takes an action at that node. The names next to the arrows are the names of the actions.)



(a) [1.5 pt] What are the subgame perfect equilibria of this game?

(b) [2 pt] What is the (4×4) payoff matrix of the corresponding bimatrix game?

(c) [1.5 pt] Find all pure Nash equilibria.

(d) [2 pt] Suppose player 1 uses the behavioral strategy b_1 with $b_1(L_1) = 1/3$, $b_1(R_1) = 2/3$, $b_1(L_4) = 1/4$ and $b_1(R_4) = 3/4$. ($b_1(X)$ denotes the behavioral probability that player 1 selects action X at the corresponding node.) What is the associated mixed strategy?

2. (a) [2 pt] Give an example of a 3×3 bimatrix game with no pure Nash equilibrium.

(b) [3 pt] Consider the 2×2 matrix game

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Prove that if this matrix game has no saddlepoints, then, without loss of generality, $a_{11} > a_{12}$, $a_{12} < a_{22}$, $a_{21} < a_{22}$, and $a_{11} > a_{21}$.

3. Let $N = \{1,2,3\}$. The game (N, v) is given by:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	3	0	2	4	7	6	10

(a) [1 pt] Compute the core $C(v)$. Express the core as convex hull of its extreme points.

(b) [1 pt] Is (N, v) convex? Is (N, v) super-additive?

(c) [1.5 pt] Compute the Shapley value $\Phi(v)$ using the characterization based on dividends.

(d) [2.5 pt] Compute the pre-nucleolus $\nu^*(v)$. Use the Kohlberg criterion to show that your answer is correct.

4. (a) [3 pt] For a game (N, v), the *upper vector* $M(v) \in \mathbb{R}^n$ is given by:
$M_i(v) = v(N) - v(N - \{i\}) \quad (i \in N)$.
The *lower vector* $m(v) \in \mathbb{R}^n$ is defined as:

$$m_i(v) = \max_{S: i \in S} \left(v(S) - \sum_{j \in S - \{i\}} M_j(v) \right) \qquad (i \in N).$$

Show that $m_i(v) \leq x_i \leq M_i(v)$, for each $x \in C(v)$ and each $i \in N$.

(b) [3 pt] A game (N, v) is called *strictly convex* if

$$v(S \cup \{i\}) - v(S) < v(T \cup \{i\}) - v(T) \quad \text{for all } i \in N \text{ and all } S \subset T \subseteq N \backslash \{i\}.$$

Show that in a strictly convex game all marginal vectors are different.

5. In the stochastic game

1 $(\frac{1}{2}, \frac{1}{2})$	2 $(0, 1)$
3 $(0, 1)$	1 $(\frac{1}{2}, \frac{1}{2})$

state 1

-1 $(\frac{1}{4}, \frac{3}{4})$

state 2

the players optimize their average rewards.

(a) [1 pt] Why is this game irreducible?

(b) [3 pt] The players 1 and 2 use stationary strategies $\mathbf{f} = ((\frac{1}{2}, \frac{1}{2}), (1))$ and $\mathbf{g} = ((\frac{2}{5}, \frac{3}{5}), (1))$ respectively. Determine the value vector $\mathbf{v}_\alpha(\mathbf{f}, \mathbf{g})$.

(c) [2 pt] Is the strategy $\mathbf{g}$ optimal for player 2?

6. (a) [2.5 pt] Mention the five main elements of a stochastic game and briefly explain these.

(b) [3.5 pt] Prove that if (π^1_*, π^2_*) and (π^1_{**}, π^2_{**}) are two equilibrium points of a zero-sum discounted stochastic game then also (π^1_*, π^2_{**}) and (π^1_{**}, π^2_*) are equilibrium points.

Total: 36 + 4 points