Exam MSc Course Game Theory (191521800)

November 8, 2012

All answers need to be motivated

1. (5 points) Consider the (symmetric) bimatrix game given by

$$(A, B) = \begin{pmatrix} 4,4 & 1,5 \\ 5,1 & 0,0 \end{pmatrix}$$

   (a) Compute all Nash equilibria of this game.

   (b) Write down all conditions that define the correlated equilibria of this game, and give an example of a correlated equilibrium of this game that is not a Nash equilibrium.

2. (4 points) Give an algorithm (in high level description) that computes a Nash equilibrium in any given 2-player bimatrix game with payoff matrices $A, B \in \mathbb{R}^{m\times n}$. Argue why your algorithm is correct. Is it a polynomial time algorithm in the input size of the problem?

   [Hint: You can use a linear programming solver as a "black box" subroutine to compute a solution of a system of linear inequalities, or decide that no such solution exists.]

3. (5 points) Consider the following three player cooperative game (N, v).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	2	5	4	15	18	12	24

   (a) Is the game a convex game? Argue.

   (b) Compute the core $C(N, v)$, and express it as convex hull of its extreme points. Is it different from the Weber set?

   (c) Compute the Shapley value $\Phi(N, v)$, and verify whether $\Phi(N, v) \in C(N, v)$.

4. (3 points) A man dies, leaving an estate of value 300.000. However, he also leaves three creditors with claims of 300.000, 200.000, and 100.000 respectively. How should the estate be divided among the creditors? To answer this question, model this situation as a cooperative game and compute the nucleolus of this game.

5. (4 points) Consider a strictly convex cooperative game (N, v), that is,

$$v(S \cup T) + v(S \cap T) > v(S) + v(T) \quad \text{for all } S, T \subseteq N\,.$$

   Recall the definition of marginal (payoff) vectors m^σ for a given permutation σ of N[1]. Show that, for strictly convex games, all marginal vectors are different. That is, show that if $\sigma \neq \sigma'$, then $m^\sigma \neq m^{\sigma'}$.

   [Hint: Consider any $\sigma \neq \sigma'$ and consider the minimal k where $\sigma(k) \neq \sigma'(k)$. Show that $m^\sigma_{\sigma(k)} \neq m^{\sigma'}_{\sigma(k)}$.]

[1] $m^\sigma_{\sigma(i)} = v(\sigma(1), \ldots, \sigma(i-1), \sigma(i)) - v(\sigma(1), \ldots, \sigma(i-1))$ for $i = 1, \ldots, n$, and for any permutation σ.

6. (3 points) For cooperative game (N, v), recall the definition of the domination core $DC(N, v) \subseteq I(N, v)$ as all efficient and individually rational payment vectors $\boldsymbol{x}$ that are non-dominated. Show that the core is a subset of the domination core, that is,

$$C(N, v) \subseteq DC(N, v) .$$

7. (6 points) In the following stochastic game with infinite horizon

state 1:

5 $(1, 0)$	2 $(1/2, 1/2)$
3 $(1/2, 1/2)$	1 $(0, 1)$

state 2:

-2 $(1/3, 2/3)$

the players optimize their average rewards.

(a) Why is this game irreducible?

(b) The players 1 and 2 use stationary strategies $\mathbf{f} = ((\frac{1}{2}, \frac{1}{2}), (1))$ and $\mathbf{g} = ((\frac{1}{2}, \frac{1}{2}), (1))$ respectively. Determine the value vector $\mathbf{v}_\alpha(\mathbf{f}, \mathbf{g})$.

(c) Is the strategy $\mathbf{g}$ optimal for player 2? Explain.

8. (3 points) Consider two firms that compete for the same market. Both firms have the possibility to adopt Dirty or Clean production methods. A Dirty production way means less production costs, a better prized product, and more profit. However, firms take a risk when producing Dirty. In the long run, at a sudden moment, the public opinion may turn against them, as the society's concern for the future may prevail. That is, Dirty firms will be banned in order to keep the world sustainable. The net effect, possibly through legislative measures, will be a bankruptcy of the firm, giving the other firm the opportunity for monopoly. All of this will occur with certain probabilities depending on society's reaction.

It may be clear from the above setting that both firms have a short-run incentive to make enough profit, but on the other hand, they definitely want to survive in the long run. Model this as a non-zero sum stochastic game with four states. Namely, in state 1 both firms are alive. In state 2 firm 1 is dead, while firm 2 is dead in state 3. In state 4, both firms are gone. Choose your own direct rewards and transition probabilities, and motivate why these are reasonable.

9. (3 points)

(a) What is the definition of an equilibrium point for non-zero sum discounted stochastic games?

(b) Let (π_*^1, π_*^2) be an equilibrium point of a discounted stochastic game. Prove that componentwise $\mathbf{v}_\beta^1(\pi_*^1, \pi_*^2) \geq \mathbf{v}_\beta^1 := \min_{\pi^2} \max_{\pi^1} \mathbf{v}_\beta^1(\pi^1, \pi^2)$.

Total: 36 + 4 = 40 points