Exam Markov Decision Theory and Algorithmic Methods (153192)

April 13, 2007

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 4 exercises with the following weights:

exercise	1	2	3	4
points	10	10	12	8

Good luck!

1. An intensively used lease-car is inspected every weekend (after being used for a week). The inspection report is summarized in a grade ranging from 1 to 5. The higher the grade, the worse the condition of the car. A perfect car receives grade 1. The grade 5 indicates that the car cannot be used any more. For the intermediate grades (2 till 4) there is a possibility to have the car repaired preventively. If this is done, the car cannot be used for a week. After the preventive repair the car is in perfect condition. A car with grade 5 should be replaced by a new one, but a new car (that is always in perfect condition) takes two weeks to arrive. The table below shows the probabilities $p(i, j)$ that a car with grade $i = 1, 2, 3, 4$ that is not being repaired receives a grade $j = 1, 2, 3, 4, 5$ after the next week:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0.75	0.20	0.05	0	0
$i = 2$	0	0.50	0.20	0.20	0.10
$i = 3$	0	0	0.50	0.25	0.25
$i = 4$	0	0	0	0.30	0.70

A car with grade 1, 2, 3 or 4 that is not being repaired, generates a profit of 1. A car that is preventively repaired, generates no profit that week. When the grade is 5, there are no profits for two weeks (lead time). The goal is to maximise the average weekly profit.

   (a) Formulate this problem as a Markov decision problem by indentifying the states, actions, immediate rewards and transition probabilities. (Hint: Introduce an extra state 0 to indicate that a new car is on order for a week already.)

   (b) Suppose one decides that the car should always be repaired preventively if its grade equals 2, 3 or 4. Calculate the average profit and the corresponding relative value vector (bias) of this policy.

   (c) Carry out one interation of the policy iteration algorithm, starting with the decision rule in (b), and determine the new policy.

   (d) Show that it is optimal to have no preventive repair at grade 2 while repairing preventively at grades 3 and 4.

2. Consider a discounted Markov decison problem (MDP) with infinite horizon.

   (a) Give a definition of this kind of MDP.

   (b) Formulate the optimality equations.

   (c) Prove that v_λ^* is the unique solution of the optimality equations.

   (d) Formulate the value-iteration algorithm for this MDP.

   (e) Prove for the value-iteration algorithm that (i) the value vector v^n generated by this algorithm converges in norm to v_λ^*, and (ii) the stationary policy $(d_\varepsilon)^\infty$ obtained in the final step of the algorithm is ε-optimal.

3. Consider the performance of a database system where there are two replications of the data. Each replication is independently accessible and modeled by a server. Requests to access to database are assumed to queue in a central location and correspond to read or write operations. To preserve the integrity of both copies of the database, we assume that write requests must wait until both copies of the database are available before beginning execution. Both copies are assumed to be updated in parallel and released simultaneously. Read requests can be processed by any copy of the database. Both types of requests are assumed to wait in the queue in the order in which they arrive. We assume that requests arrive to the system from a Poisson point source with intensity of λ and that the probability a given request is a read (resp., write) is given by r (resp., $1-r$). Service times for both read and write requests are assumed to be exponential with an average value of μ^{-1}. Since we assume that writes are served in parallel, the total service time for write requests equals the maximum of two exponential random variables with parameter μ. We will model the system in matrix geometric form.

   (a) Give the diagram with transitions and transition rates for this queue, and specify the levels and phases.

   (b) Provide a description in matrix geometric form, that is specify the matrices A_0, A_1, A_2.

   (c) Give the stability condition explicitly in terms of the matrices A_0, A_1, A_2. From this result, now give the stabilty condition in terms of the parameters λ, μ, r.

   (d) Give the matrix equation for the rate matrix.

   (e) Can this matrix equation be solved via the balance principle?

   (f) Provide an explicit expression for the rate matrix.

   (g) Give an interpretation for the elements of the rate matrix.

   (h) Give an explicit expression for the equilibrium distribution (that is also specify π_0).

4. Consider a network of two infinite server queues with exponential service requirement with rate μ_i at queue i, $i = 1, 2$, that is when the number of customers in queue i equals n_i, the departure rate from queue i equals $n_i \mu_i$. A customer departing from queue 1 moves to queue 2 with probability p_{12}, and a customer departing from queue 2 moves to queue 1 with probability p_{21}. Customers arrive to the system according to a Poisson process with rate λ_i to queue i, $i = 1, 2$. Let $N = \{N(t),\ t \geq 0\}$ record the number of customers at the two queues.

   (a) Show that N is a Markov chain, and specify its state space and transitions rates.

   (b) Give the stability condition for N.

   (c) Give the global balance equations for N.

   (d) For a general Markov chain (that is not specifically for N), provide the a description of the uniformized Markov chain (give the transition probabilities, and derive an expression for the state probabilities).

   (e) Can N be uniformized?

   (f) Again, consider a general Markov chain that can be uniformized. The expression $P(t) = \mathrm{e}^{-\Lambda t} \sum_{n=0}^{\infty} \frac{(\Lambda t)^n}{n!} P(0) R^n$ for the state probabilities obtained via uniformization is not amenable for numerical evaluation.
   Show why truncation (replace $\sum_{n=0}^{\infty}$ by the finite sum $\sum_{n=0}^{K}$) will not lead to a good approximation for all t not even when K is chosen arbitrarily large. Indicate an improvement that does lead to a good approximation for all t when K is chosen large enough (include proof).