Kenmerk: SK-EWI06/TW/SK/019/EvD

Datum: 27 januari 2006

Exam Measure and Probability (157040)

Friday, 3 February 2006, 9.00 - 12.00 p.m.

This exam consists of 8 problems

1. Let Ω be a set, $\mathcal{F}$ a σ-field of subsets of Ω, and $\mu : \mathcal{F} \to \mathbb{R}$. When do we call
   a. μ an outer measure?
   b. μ a measure?
   c. $(\Omega, \mathcal{F}, \mu)$ a probability space?
2. a. Define what is meant by saying that $f : \mathbb{R} \to \mathbb{R}$ is (Lebesgue) measurable.
   b. Show that the indicator function of a set $A \subset \mathbb{R}$ (defined by $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ otherwise), is measurable if and only if A is a measurable set.
   c. Give an example of a non-measurable function f such that $|f|$ is measurable.
3. Consider the probability space $([0,1], \mathcal{B}_{[0,1]}, m_{[0,1]})$, and $X : \; [0,1] \to \mathbb{R}$.
   a. Under which condition is the function X a random variable?
   b. If X is a random variable, how is its *probability distribution* P_X defined?
   c. Express P_X in terms of Lebesgue measure m if X is given by $X(\omega) := 3\omega - 2, \; 0 \leq \omega \leq 1$.
4. Suppose that f is a Lebesgue-measurable function such that $f \geq 0$.
   a. How is $\int_{\mathbb{R}} f dm$ defined?
   b. Show that $\int_{\mathbb{R}} f dm > 0$ if $m(\{x : f(x) > 0\}) > 0$.
5. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.
   a. State the *dominated convergence theorem.*
   b. Evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{1}{(1+\frac{x}{n})^n \sqrt[n]{x}} dx.$$

6. Consider the probability space $([0,1], \mathcal{B}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of

   a. the random variable X, given by $X(\omega) = 3\omega - 2$,

   b. the random variable X given by $X(\omega) = \min(\omega, 1-\omega)$.

7. Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with joint density

$$f_{X,Y}(x,y) = \mathbf{1}_A(x,y), \quad (x,y) \in \mathbb{R}^2,$$

   where A is the triangle with corners at (0,0), (1,0) and (0,2).

   a. Find $P(X > Y)$.

   b. Find the conditional density $f_{Y|X}(y|X=x)$ of Y given $X = x$.

   c. Determine $E(Y|X)$.

8. Consider the probability space $((0,1), \mathcal{B}_{(0,1)}, m_{(0,1)})$ and, for $n = 1, 2, \ldots$, set $X_n(\omega) = \sqrt{n}(1-\omega)^n$, $0 \leq \omega < 1$. Which of the following statements are true? (Justify your answers).

   a. $X_n \to 0$ in probability.

   b. $X_n \to 0$ almost surely.

   c. $X_n \to 0$ pointwise.

   d. $X_n \to 0$ in L^1-norm.

   e. $X_n \to 0$ in L^2-norm.

1	2	3	4	5	6	7	8
3	3	3	3	3	4	3	3

Mark: $\frac{\text{Total}}{25} \times 9 + 1$ (rounded)