Datum: 7 februari 2007

Exam Measure and Probability (157040)
Friday, 8 February 2007, 13.30 - 16.30 p.m.

This exam consists of 8 problems

1. Let Ω be a set, $\mathcal{F}$ a σ-field of subsets of Ω, and $\mu : \mathcal{F} \to \mathbb{R}$ a function. When do we call

   a. μ an outer measure?

   b. μ a measure?

   c. $(\Omega, \mathcal{F}, \mu)$ a probability space?

2. Suppose $E \subset \mathbb{R}$ is a (Lebesgue-)measurable set.

   a. Define what is meant by saying that $f : E \to \mathbb{R}$ is measurable.

   b. Show that $f : E \to \mathbb{R}$ is measurable if and only if $\{x \in E : f(x) > r\}$ is measurable for each rational number r.

   c. Show that $\{x \in E : f_1(x) > f_2(x)\}$ is measurable if $f_1 : E \to \mathbb{R}$ and $f_2 : E \to \mathbb{R}$ are measurable. (Do not use any result without proof other than b.)

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$. Let $f : \mathbb{R} \to \mathbb{R}$ and $f \geq 0$, and define $\nu : \mathcal{M} \to \mathbb{R}$ by

$$\nu(A) = \int_A f dm, \quad A \in \mathcal{M}.$$

   a. State the *monotone convergence theorem.*

   b. Show that ν is a measure. (Hint: use the monotone convergence theorem.)

4. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *dominated convergence theorem.*

   b. Evaluate

$$\lim_{n\to\infty} \int_0^1 e^{-nx^2} dx.$$

5. Show that the function $f(x) = x^{-1} \sin x$ is not Lebesgue integrable over the interval (π, ∞).

6. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of

   a. the random variable X, given by $X(\omega) = 2\omega - 1$,

   b. the random variable X given by $X(\omega) = \max(\omega, 1-\omega)$.

7. Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with joint density

$$f_{X,Y}(x,y) = \mathbf{1}_A(x,y), \quad (x,y) \in \mathbb{R}^2,$$

   where A is the triangle with corners at (0,2), (1,0) and (1,2).

   a. Find $P(X > Y)$.

   b. Find the conditional density $f_{X|Y}(x|Y=y)$ of X given $Y = y$.

   c. Determine $E(X|Y)$.

8. Consider the probability space $((0,1), \mathcal{M}_{(0,1)}, m_{(0,1)})$ and, for $n = 1, 2, \ldots$, set

$$X_n(\omega) = \begin{cases} n & \text{if } \ 0 < \omega < \frac{1}{n} \\ 0 & \text{if } \ \frac{1}{n} < \omega < 1. \end{cases}$$

   Which of the following statements are true? (Justify your answers).

   a. $X_n \to 0$ in probability.

   b. $X_n \to 0$ almost surely.

   c. $X_n \to 0$ pointwise.

   d. $X_n \to 0$ in L^1-norm.

   e. $X_n \to 0$ in L^2-norm.

1	2	3	4	5	6	7	8
3	4	3	3	2	3	3	3

Mark: $\frac{\text{Total}}{24} \times 9 + 1$ (rounded)