Kenmerk: EWI10/TW/SP/026
Datum: 22 april 2010

Exam Measure and Probability (157040)
Thursday, 29 April 2010, 13.45 - 16.45 p.m.

This exam consists of 6 problems

1. a. Define what is meant by $m^*(A)$, the Lebesgue outer measure of $A \subset \mathbb{R}$.

   b. Use the countable subadditivity (and the definition) of Lebesgue outer measure to show that $m^*(A) = 0$ implies $m^*(A \cup B) = m^*(B)$ for each $B \subset \mathbb{R}$.

   c. Define what is meant by saying that $A \subset \mathbb{R}$ is (Lebesgue) measurable.

   d. Define what is meant by saying that $f : \mathbb{R} \to \mathbb{R}$ is (Lebesgue) measurable.

   e. Show that the indicator function of a set $A \subset \mathbb{R}$ (defined by $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ otherwise), is (Lebesgue) measurable if and only if A is a (Lebesgue) measurable set.

   f. Define what is meant by saying that the (Lebesgue) measurable function $f : \mathbb{R} \to \mathbb{R}$ is (Lebesgue) integrable.

2. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *monotone convergence theorem.*

   b. (*Borel-Cantelli lemma*) Suppose $\{E_k\}$ is a sequence of measurable sets satisfying
   $$\sum_{k=1}^{\infty} m(E_k) < \infty.$$
   Show that $m(F) = 0$ when $F = \{x : x \text{ belongs to infinitely many sets } E_k\}$. (Hint: define $f_n(x) = \sum_{k=1}^{n} \mathbb{I}_{E_k}(x)$.)

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *dominated convergence theorem.*

   b. Evaluate
   $$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} dx.$$
   (Apart from a normalizing constant, the integrand is the density function for the t-distribution with n degrees of freedom.)

4. Consider the (Lebesgue) measurable function $f : \mathbb{R}^2 \to \mathbb{R}$.
   a. What does *Fubini's theorem* tell us about $\iint_{\mathbb{R}^2} f dm_2$?
   b. Show that if f is the joint density function of the absolutely continuous random variables X and Y, then X and Y are independent if and only if
$$f(x, y) = f_X(x) f_Y(y) \text{ a.e.}$$

5. Let μ and ν be two finite measures on a measurable space $(\Omega, \mathcal{F})$.
   a. What is meant by $\mu(A) \ll \nu(A)$ (μ is *absolutely continuous with respect to* ν)?

   Suppose that, for some $a > 0,\ b > 0$, we have $a\mu(A) \leq \nu(A) \leq b\mu(A)$ for all $A \in \mathcal{F}$.

   b. Show that μ and ν are equivalent measures (that is, $\mu \ll \nu$ and $\nu \ll \mu$).
   c. Show that the respective Radon-Nikodym derivatives $f = d\nu/d\mu$ and $g = d\mu/d\nu$ satisfy $a \leq f \leq b$ μ-a.e. and $b^{-1} \leq g \leq a^{-1}$ ν-a.e.

6. Consider the probability space $([0, 1], \mathcal{M}_{[0,1]}, m_{[0,1]})$ and, for $n = 1, 2, \ldots$, set
$$X_n(\omega) = \begin{cases} n^{2/3} & \text{if } \ 0 \leq \omega < \frac{1}{n} \\ n^{-1/3} & \text{if } \ \frac{1}{n} \leq \omega \leq 1. \end{cases}$$
   a. Find the distribution function $F_n(x)$ of X_n and $\mathbb{E}(X_n)$.
   b. Which of the following statements are true? (Justify your answers).
      (i) $X_n \to 0$ in probability.
      (ii) $X_n \to 0$ almost surely.
      (iii) $X_n \to 0$ pointwise.
      (iv) $X_n \to 0$ in L^1-norm.
      (v) $X_n \to 0$ in L^2-norm.
      (vi) $X_n \to 0$ uniformly.

1	2	3	4	5	6
7	3	4	5	4	4

Mark: $\frac{\text{Total}}{27} \times 9 + 1$ (rounded)