Exam Measure and Probability (191570401)

This exam consists of 7 problems

1. Consider the measure space $((0,1), \mathcal{M}_{(0,1)}, m_{(0,1)})$.

   a. Define what is meant by saying that $f : (0,1) \to \mathbb{R}$ is measurable.

   b. Define what is meant by saying that $f : (0,1) \to \mathbb{R}$ is integrable.

   A measurable function $f : (0,1) \to \mathbb{R}$ is said to be *mean-square integrable* if $\int_{(0,1)} f^2 dm < \infty$.

   c. Show that every mean-square integrable function is integrable.

2. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *monotone convergence theorem*.

   b. (*Borel-Cantelli lemma*) Suppose $\{E_k\}$ is a sequence of measurable sets satisfying
   $$\sum_{k=1}^{\infty} m(E_k) < \infty.$$
   Show that $m(F) = 0$ when $F = \{x : x \text{ belongs to infinitely many sets } E_k\}$. (Hint: A possible approach is to define $f_n = \sum_{k=1}^{n} \mathbb{I}_{E_k}$, $f = \lim_{n\to\infty} f_n$, and show that $\int_{\mathbb{R}} f dm < \infty$.)

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *dominated convergence theorem*.

   b. Evaluate
   $$\lim_{n\to\infty} \int_0^{\infty} \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx.$$

4. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of

   a. $X : [0,1] \to \mathbb{R}$ given by $X(\omega) = \min\{\omega, 1-\omega\}$ (the distance to the nearest endpoint of the interval $[0,1]$);

   b. $X : [0,1]^2 \to \mathbb{R}$, the distance to the nearest edge of the square $[0,1]^2$.

5. Consider the (Lebesgue) measurable function $f : \mathbb{R}^2 \to \mathbb{R}$.

   a. What does *Fubini's theorem* tell us about $\iint_{\mathbb{R}^2} f dm_2$?

   b. Evaluate

   $$\int_E y \sin(x) e^{-xy} dx dy,$$

   where $E = (0, \infty) \times (0, 1)$, and justify your steps.

6. Consider the interval $[-1, 1]$ with Lebesgue measure $m_{[-1,1]}$. and let ν be a measure on the measurable space $([-1, 1], \mathcal{B}_{[-1,1]})$ such that

   $$\nu([-1, x]) = \begin{cases} 0 & \text{if } \; -1 \leq x < 0 \\ 1 + x^2 & \text{if } \; 0 \leq x \leq 1. \end{cases}$$

   a. Show that ν is *not* absolutely continuous with respect to $m_{[-1,1]}$.

   b. Give the Lebesgue decomposition of ν with respect to $m_{[-1,1]}$, that is, determine ν_a and ν_s such that $\nu = \nu_a + \nu_s$, $\nu_a \ll m_{[-1,1]}$ and $\nu_s \perp m_{[-1,1]}$.

   c. Determine the Radon-Nikodym derivative of ν_a with respect to $m_{[-1,1]}$.

7. Consider the probability space $([0, 1], \mathcal{M}_{[0,1]}, m_{[0,1]})$ and set

   $$X_n(\omega) = \max \left\{ n - n^2 |\omega - \frac{1}{n}|, 0 \right\}, \quad n = 1, 2, \ldots.$$

   a. Does X_n converge to 0 uniformly? Pointwise?

   b. Does X_n converge to 0 almost surely? In probability?

   c. Does X_n converge to 0 in L^1-norm?

   Motivate your answers.