Exam Measure and Probability (191570401)
Monday 21 January 2013, 8.45 - 11.45 p.m.

This exam consists of 8 problems

1. Let Ω be a set, $\mathcal{F}$ a collection of subsets of Ω, and $\mu : \mathcal{F} \to \mathbb{R}$. When do we call
   a. $\mathcal{F}$ a σ-field?
   b. μ an outer measure?
   c. μ a measure?
   d. $(\Omega, \mathcal{F}, \mu)$ a probability space?

2. Suppose $E \subset \mathbb{R}$ is a (Lebesgue-)measurable set.
   a. Define what is meant by saying that $f : E \to \mathbb{R}$ is measurable.
   b. Show that $f : E \to \mathbb{R}$ is measurable if and only if $\{x \in E : f(x) > r\}$ is measurable for each rational number r. (Hint: for each $a \in \mathbb{R}$ there is a decreasing sequence of rational numbers r_n such that $a = \lim_{n\to\infty} r_n$.)

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.
   a. State the *monotone convergence theorem*.

   Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and $f \geq 0$, and define $\nu : \mathcal{M} \to \mathbb{R}$ by
   $$\nu(A) = \int_A f dm, \quad A \in \mathcal{M}.$$
   b. Show that ν is a measure. (Hint: use the monotone convergence theorem.)

4. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.
   a. State the *dominated convergence theorem*.
   b. Evaluate
   $$\lim_{n\to\infty} \int_0^1 \frac{n \sin(x)}{1 + n^2\sqrt{x}} dx.$$

5. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of
   a. the random variable X, given by $X(\omega) = 2\omega - 1$,
   b. the random variable X given by $X(\omega) = \max(\omega, 1 - \omega)$.

6. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces, and let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$.
   a. (*Fubini's Theorem*) Under which condition do we have
$$\int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2) = \int_{\Omega_1} \left(\int_{\Omega_2} f d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} f d\mu_1 \right) d\mu_2 ?$$
   b. Evaluate
$$\int_E y \sin x e^{-xy} dx dy,$$
   where $E = (0, \infty) \times (0, 1)$, and justify your steps.

7. Let $(\Omega, \mathcal{F})$ be a measurable space and let μ and ν be finite measures on $(\Omega, \mathcal{F})$.
   a. What is meant by $\nu \ll \mu$ (ν is *absolutely continuous with respect to* μ)?
   b. What does the *Radon-Nikodym Theorem* say about the relation between μ and ν if $\nu \ll \mu$?
   c. Give the *Radon-Nikodym derivative* $\frac{d\nu}{d\mu}$ if $\nu \ll \mu$ and Ω is finite.

8. Consider the probability space $([0, 1], \mathcal{M}_{[0,1]}, m_{[0,1]})$ and set
$$X_n(\omega) = \max \left\{ n - n^2 |\omega - \frac{1}{n}|, 0 \right\}, \quad n = 1, 2, \ldots .$$
   a. Does X_n converge to 0 uniformly? Pointwise?
   b. Does X_n converge to 0 almost surely? In probability?
   c. Does X_n converge to 0 in L^1-norm?

   Motivate your answers.

1				2		3		4		5		6		7			8		
a	b	c	d	a	b	a	b	a	b	a	b	a	b	a	b	c	a	b	c
2	2	2	1	1	3	1	2	2	2	2	2	2	2	1	1	2	2	2	2

Mark: $\frac{\text{Total}}{36} \times 9 + 1$ (rounded)