UNIVERSITEIT TWENTE

Faculteit Elektrotechniek, Wiskunde en Informatica

Exam Measurability and Probability (1915703401) on Monday, January 23, 2017, 8:45 - 11:45 hours.

The solutions of the exercises should be clearly formulated and clearly written down. Moreover, you should in all cases include a convincing argument with your answer.
With this exam a calculator is **not** permitted. Also a formula sheet is **not** permitted.

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1. Consider the measurable space $(\Omega, \mathcal{B}, m)$ with $\Omega = [-1,1]$, $\mathcal{B}$ is the collection of Borel sets and m is the Lebesgue measure. Consider the three functions: $f_1, f_2, f_3 : \Omega \to \mathbb{R}$ defined by:

$$f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = x^3$$

Let $\mathcal{F}_1$ be the smallest σ-algebra such that f_1 as a mapping from $(\Omega, \mathcal{F}_1)$ to $(\Omega, \mathcal{B})$ is measurable. We define $\mathcal{F}_2$ and $\mathcal{F}_3$ similarly as the smallest σ-algebras such that f_2 and f_3 respectively are measurable.

a) Verify whether $\mathcal{F}_1 \subset \mathcal{F}_2$, $\mathcal{F}_1 = \mathcal{F}_2$ or $\mathcal{F}_1 \supset \mathcal{F}_2$.

b) Verify whether $\mathcal{F}_1 \subset \mathcal{F}_3$, $\mathcal{F}_1 = \mathcal{F}_3$ or $\mathcal{F}_1 \supset \mathcal{F}_3$.

c) Verify whether $\mathcal{F}_1 \subset \mathcal{B}$, $\mathcal{F}_1 = \mathcal{B}$ or $\mathcal{F}_1 \supset \mathcal{B}$.

2. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$. Investigate the convergence of:

$$\lim_{n\to\infty} \int_1^e \frac{n}{(x+n)x^{1+1/n}}\, dm$$

If you use any theorems from the book then clearly formulate that theorem.

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$. Investigate the convergence of:

$$\lim_{n\to\infty} \int_0^n \frac{1}{1+nx\ln x}\, dm$$

If you use any theorems from the book then clearly formulate that theorem.

4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $\Omega = \cup_{n=1}^{\infty} E_n$ where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\mu(E_n) < \infty$ for all $n \geq 1$. Define ν on $\mathcal{F}$ by:

$$\nu(F) = \sum_{n=1}^{\infty} 2^{-n} \frac{\mu(F \cap E_n)}{\mu(E_n)+1}$$

a) Prove that ν is a measure on $\cancel{(\Omega, \mathcal{B})}$. $(\Omega, \mathcal{F})$

b) Let $F \in \mathcal{F}$. Show that $\mu(F) = 0$ if and only if $\nu(F) = 0$.

c) Find explicitly two positive integrable functions f and g such that:

$$\nu(A) = \int_A f\, d\mu, \qquad \mu(A) = \int_A g\, d\nu.$$

for all $A \in \mathcal{F}$.

see reverse side

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5. Consider the probability space $(\Omega, \mathcal{F}, \mu)$ with $\Omega = [0,2]$, $\mathcal{F} = \mathcal{M}_{[0,2]}$ and $\mu = \frac{1}{2}m_{[0,2]}$. Let X and Y be random variables on the product space $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$ defined by:

$$X(\omega_1, \omega_2) = \omega_1\omega_2, \qquad Y(\omega_1, \omega_2) = 2\omega_1^2$$

   a) Find the (cumulative) distribution function F_X
   b) Compute $\mathbb{E}(X)$
   c) Compute $P(X > Y)$
   d) Compute $\mathbb{E}(X \mid Y)$

6. Consider the probability space $(\Omega, \mathcal{F}, \mu)$ with $\Omega = [0, \infty)$, $\mathcal{F} = \mathcal{M}_{[0,\infty)}$. Let μ be such that

$$\mu([a,b]) = \int_a^b \frac{1}{(x+1)^2} dm$$

   Define a sequence of random variables:

$$X_n(\omega) = \max\{0, n^2 - n^5|\omega - n|\}$$

   Which of the following statements are true? (Justify your answers).

   a) $X_n \to 0$ in probability.
   b) $X_n \to 0$ almost surely.
   c) $X_n \to 0$ pointwise.
   d) $X_n \to 0$ in L_1-norm.
   e) $X_n \to 0$ in L_2-norm.

For the questions the following number of points can be awarded:

Exercise	1.	9 points	Exercise	4.	9 points
Exercise	2.	9 points	Exercise	5.	9 points
Exercise	3.	9 points	Exercise	6.	9 points

The final grade is determined by adding 6 points to the total number of points awarded and dividing by 6.