

Exam Measure and Probability (157040)
Thursday, 12 April 2007, 9.00 - 12.00 a.m.

This exam consists of 7 problems

1.
 - a. Define what is meant by $m^*(A)$, the Lebesgue outer measure of a subset A of \mathbb{R} .
 - b. Define what is meant by saying that $A \subset \mathbb{R}$ is measurable.
 - c. Show that if $A \subset \mathbb{R}$ and $m^*(A) = 0$, then A is measurable.
2. Suppose $E \subset \mathbb{R}$ is a (Lebesgue-)measurable set.
 - a. Define what is meant by saying that $f : E \rightarrow \mathbb{R}$ is measurable.
 - b. Show that $f : E \rightarrow \mathbb{R}$ is measurable if and only if $\{x \in E : f(x) > r\}$ is measurable for each rational number r . (Hint: for each $a \in \mathbb{R}$ there is a decreasing sequence of rational numbers r_n such that $a = \lim_{n \rightarrow \infty} r_n$.)
 - c. Show that $\{x \in E : f_1(x) > f_2(x)\}$ is measurable if $f_1 : E \rightarrow \mathbb{R}$ and $f_2 : E \rightarrow \mathbb{R}$ are measurable. (Do not use any other result than b. without proof.)
3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f \geq 0$, and define $\nu : \mathcal{M} \rightarrow \mathbb{R}$ by
$$\nu(A) = \int_A f dm, \quad A \in \mathcal{M}.$$
 - a. State the *monotone convergence theorem*.
 - b. Show that ν is a measure. (Hint: use the monotone convergence theorem.)
4. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.
 - a. State the *dominated convergence theorem*.
 - b. Evaluate
$$\lim_{n \rightarrow \infty} \int_0^1 e^{-nx^2} dx.$$
5. Consider the probability space $([0, 1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X , the distribution function, and $\mathbb{E}(X)$, the expectation of
 - a. the random variable X , given by $X(\omega) = 2\omega - 1$,

- b. the random variable X given by $X(\omega) = \max(\omega, 1 - \omega)$.
6. Let X and Y be two random variables defined on the probability space (Ω, \mathcal{F}, P) with joint density

$$f_{X,Y}(x, y) = \mathbf{1}_A(x, y), \quad (x, y) \in \mathbb{R}^2,$$

where A is the triangle with corners at $(0,2)$, $(1,0)$ and $(1,2)$.

- a. Find $P(X > Y)$.
- b. Find the conditional density $f_{X|Y}(x|Y = y)$ of X given $Y = y$.
- c. Determine $E(X|Y)$.
7. Consider the probability space $((0, 1), \mathcal{M}_{(0,1)}, m_{(0,1)})$ and, for $n = 1, 2, \dots$, set

$$X_n(\omega) = \begin{cases} n & \text{if } 0 < \omega < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < \omega < 1. \end{cases}$$

Which of the following statements are true? (Justify your answers).

- a. $X_n \rightarrow 0$ in probability.
- b. $X_n \rightarrow 0$ almost surely.
- c. $X_n \rightarrow 0$ pointwise.
- d. $X_n \rightarrow 0$ in L^1 -norm.
- e. $X_n \rightarrow 0$ in L^2 -norm.

1	2	3	4	5	6	7
3	3	2	2	2	3	3

Mark: $\frac{\text{Total}}{18} \times 9 + 1$ (rounded)