Reference: EWI09/TW/SP/005/EvD
Date: 8 January 2009

Exam Measure and Probability (157040)
Monday, 19 January 2009, 9.00 - 12.00 a.m.

This exam consists of 8 problems

1. Let Ω be a non-empty set.

   a. Define what is meant by a *σ-field* of subsets of Ω.

   b. Let $\mathcal{F}$ be a σ-field of subsets of Ω and let $B \subseteq \Omega$. Show that $\mathcal{G} = \{A \in \mathcal{F} : A \subseteq B \text{ or } B^c \subseteq A\}$ is a σ-field.

   c. Let $f : \Omega \to \mathbb{R}$ be a function and let $\mathcal{F}$ be a σ-field. What does it mean to say that f is *$\mathcal{F}$-measurable*.

2. Let Ω be a set, $\mathcal{F}$ a σ-field of subsets of Ω, and $\mu : \mathcal{F} \to [0, \infty]$ a function.

   a. Under which conditions on μ do we call $(\Omega, \mathcal{F}, \mu)$ a measure space?

   b. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that $\mu(\cup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu(A_n)$ for any sequence of sets $A_n \in \mathcal{F}$. Show by examples that the inequality may be strict or that it may be the equality.

3. Two players (A and B) play the following game. Each player rolls a fair die and they write down the result: say A rolls n_A and B rolls n_B. If n_A is even, A pays B €n_B; if n_A is odd, B pays A €n_A (we think of A paying B a negative amount). Describe a probability space modelling this game, on which the amount that A pays B is a random variable X. Find the expectation of X.

4. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$ and let $E \in \mathcal{M}$. Let $f : E \to \mathbb{R}$ be a nonnegative measurable function, and $\{s_n\}_{n\ge1}$ a sequence of nonnegative, simple functions that *decreases* monotonically to f on E pointwise.

   a. State the *monotone convergence theorem.*

   b. Show, by using the monotone convergence theorem, that
   $$\lim_{n\to\infty} \int s_n dm = \int f dm.$$

5. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *dominated convergence theorem*.

   b. Evaluate
   $$\lim_{n\to\infty} \int_0^1 e^{-nx^2}\,dx.$$

6. Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with joint density
$$f_{X,Y}(x,y) = \mathbf{1}_A(x,y), \quad (x,y) \in \mathbb{R}^2,$$
where A is the triangle with corners at (0,2), (1,0) and (1,2).

   a. Find $P(X > Y)$.

   b. Find the conditional density $f_{X|Y}(x|Y = y)$ of X given $Y = y$.

   c. Determine $E(X|Y)$.

7. Let $(\Omega, \mathcal{F})$ be a measurable space and let μ and ν be finite measures on $(\Omega, \mathcal{F})$.

   a. What is meant by $\nu \ll \mu$ (ν is *absolutely continuous with respect to* μ)?

   b. What does the *Radon-Nikodym Theorem* say about the relation between μ and ν if $\nu \ll \mu$?

   c. Give the *Radon-Nikodym derivative* $\frac{d\nu}{d\mu}$ if $\nu \ll \mu$ and Ω is finite.

8. Consider a sequence of functions $f_n(x) = n^2 e^{-n|x|}$, $x \in \mathbb{R}$, and let $f(x) = 0$, $x \in \mathbb{R}$. Does f_n converge to f

   a. uniformly on $\mathbb{R}$?

   b. pointwise?

   c. almost everywhere?

   d. in L^p-norm ($p \geq 1$)?

points:

1	2	3	4	5	6	7	8
5	4	3	3	3	3	5	4

mark: $\frac{\text{total}}{30} \times 9 + 1$ (rounded)