Exam Measure and Probability (191570401)
Monday 20 January 2014, 8.45 - 11.45 p.m.

This exam consists of 7 problems

1. Let Ω be a set, $\mathcal{F}$ a collection of subsets of Ω, and $\mu : \mathcal{F} \to [0, \infty)$. When do we call

   a. $\mathcal{F}$ a σ-field?

   b. μ an outer measure?

   c. μ a measure?

   Suppose $\mathcal{F}$ is a σ-field and μ a *finitely-additive* set function, that is, a function such that $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever A and B are disjoint sets in $\mathcal{F}$. Also suppose μ has the following property: If $E_1 \supset E_2 \supset E_3 \supset \ldots$ are sets in $\mathcal{F}$ such that $\cap_i E_i = \emptyset$, then $\lim_{i\to\infty} \mu(E_i) = 0$.

   d. Prove that μ is a measure on $(\Omega, \mathcal{F})$.

2. Consider the measure space $((0,1), \mathcal{M}_{(0,1)}, m_{(0,1)})$.

   a. What is meant by saying that $f : (0,1) \to \mathbb{R}$ is measurable?

   b. What is meant by saying that $f : (0,1) \to \mathbb{R}$ is integrable?

   A measurable function $f : (0,1) \to \mathbb{R}$ is said to be *mean-square integrable* if $\int_{(0,1)} f^2 dm < \infty$.

   c. Show that every mean-square integrable function is integrable.

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *monotone convergence theorem.*

   b. (*Borel-Cantelli Lemma*) Suppose $\{E_k\}$ is a sequence of measurable sets satisfying
   $$\sum_{k=1}^{\infty} m(E_k) < \infty.$$
   Show that $m(F) = 0$ when $F = \{x : x \text{ belongs to infinitely many sets } E_k\}$. (Hint: Define $f_n = \sum_{k=1}^{n} \mathbb{I}_{E_k}$, $f = \lim_{n\to\infty} f_n$, and show that $\int_{\mathbb{R}} f dm < \infty$.)

4. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *dominated convergence theorem.*

   b. Evaluate
   $$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} dx.$$

5. Consider the (Lebesgue) measurable function $f : \mathbb{R}^2 \to \mathbb{R}$.

   a. What does *Fubini's theorem* tell us about $\int_{\mathbb{R}^2} f dm_2$?

   b. Show that if f is the joint density function of the absolutely continuous random variables X and Y, then X and Y are independent if and only if
   $$f(x,y) = f_X(x) f_Y(y) \text{ a.e.}$$

6. Let $\mu_i,\ i = 1,2,3$ be finite measures on a measurable space $(\Omega, \mathcal{F})$.

   a. What is meant by $\mu_1 \ll \mu_2$ (μ_1 is *absolutely continuous with respect to* μ_2)?

   b. What does the *Radon-Nikodym Theorem* say about the relation between μ_1 and μ_2 if $\mu_1 \ll \mu_2$?

   c. Let $\mu_1 = \delta_0 + \delta_1, \mu_2 = m_{[0,1]}$ and $\mu_3 = \mu_1 + \mu_2$. For which $i \neq j$ do we have $\mu_i \ll \mu_j$? Find the *Radon-Nikodym derivative* in each such case.

7. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$ and set
   $$X_n(\omega) = \max\left\{ n - n^2|\omega - \frac{1}{n}|, 0 \right\}, \qquad n = 1,2,\ldots.$$

   a. Does X_n converge to 0 uniformly? Pointwise?

   b. Does X_n converge to 0 almost surely? In probability?

   c. Does X_n converge to 0 in L^1-norm?

   Motivate your answers.

1				2			3		4		5		6			7			Σ
a	b	c	d	a	b	c	a	b	a	b	a	b	a	b	c	a	b	c	
2	2	2	3	1	1	2	2	3	2	2	2	3	1	2	2	2	1	1	36

Mark: $\frac{\text{Total}}{36} \times 9 + 1$ (rounded)