Exam Queueing Theory
Tuesday, May 31, 2016, 13.30 – 16.30.

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1. Jobs arrive at a rate of $\lambda = 8$ jobs per hour and jobs are processed at rate $\mu = 14$ jobs per hour. So $\rho = \frac{\lambda}{\mu} = \frac{4}{7}$.

   a) $$E(S) = \frac{\frac{1}{\mu}}{1-\rho} = \frac{\frac{1}{14}}{1-\frac{4}{7}} = \frac{1}{6} = 0.167 \text{ hours.}$$

   b) $$P(S > t) = e^{-\mu(1-\rho)t},$$
   and thus
   $$P\left(S > \frac{2}{\mu(1-\rho)}\right) = e^{-2} = 0.135.$$

   c)
   $$\begin{aligned} p_0 8 &= p_{1,f}14 + p_{1,s}2, \\ p_{1,f}22 &= p_0 8 + p_2 2, \\ p_{1,s}10 &= p_2 14, \\ p_2 24 &= p_{1,f}8 + p_{1,s}8 + p_3 16, \\ p_n 24 &= p_{n-1}8 + p_{n+1}16, \quad n \geq 3. \end{aligned}$$

   d) $$E(L) = (p_{1,f} + p_{1,s}) + \sum_{n=2}^{\infty} n p_n = \frac{10}{27} + \sum_{n=2}^{\infty} n \frac{5}{27}\left(\frac{1}{2}\right)^{n-2} = \frac{40}{27}$$
   and thus with Little,
   $$E(S) = \frac{E(L)}{\lambda} = \frac{5}{27} > \frac{1}{6}.$$

2. Customer orders arrive at a rate of $\lambda = \frac{1}{4}$ orders per hour. The production of a product is $B = E + 1$ hours, where E is exponentially distributed with a mean of 2 hours. So $\rho = \lambda E(B) = \frac{1}{4} \cdot 3 = \frac{3}{4}$.

   a) For $0 \leq t < 1$,
   $$F_B(t) = P(B \leq t) = 0$$
   and for $t \geq 1$,
   $$F_B(t) = P(E + 1 \leq t) = P(E \leq t-1) = 1 - e^{-\frac{1}{2}(t-1)}.$$
   For the Laplace-Stieltjes transform we have
   $$\widetilde{B}(s) = E(e^{-sB}) = E(e^{-s(E+1)}) = e^{-s}E(e^{-sE}) = \frac{e^{-s}}{1+2s}.$$

b) For $t \geq 0$,

$$f_R(t) = \frac{1 - F_B(t)}{E(B)} = \frac{1 - F_B(t)}{3}.$$

Hence, for $0 \leq t < 1$

$$f_R(t) = \frac{1}{3}$$

and for $t \geq 1$,

$$f_R(t) = \frac{1}{3} e^{-\frac{1}{2}(t-1)} = \frac{2}{3} \cdot \frac{1}{2} e^{-\frac{1}{2}(t-1)}.$$

For the Laplace-Stieltjes transform we get

$$\widetilde{R}(s) = \frac{1 - \widetilde{B}(s)}{sE(B)} = \frac{1 - \frac{e^{-s}}{1+2s}}{3s} = \frac{1 + 2s - e^{-s}}{3s(1+2s)}.$$

c) $$\widetilde{W}(s) = \frac{1 - \rho}{1 - \rho \widetilde{R}(s)} = \frac{\frac{1}{4}}{1 - \frac{3}{4} \frac{1+2s-e^{-s}}{3s(1+2s)}} = \frac{s(1+2s)}{(4s-1)(1+2s) + e^{-s}}.$$

d) Note that $E(R) = \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot 3 = 2\frac{1}{6}$, so

$$E(W) = \frac{\rho E(R)}{1 - \rho} = \frac{13}{2} = 6\frac{1}{2} \text{ hours.}$$

3. a) $\lambda < 2\mu$.

b) The (global) balance equations for the set $\{0, 1, \ldots, n\}$ are

$$p_n \lambda = (p_{n+1} + p_{n+2})\mu, \quad n \geq 0.$$

For $\lambda = 10$ and $\mu = 9$ and substituting $p_n = Cx^n$ we get

$$10Cx^n = 9Cx^{n+1}(1 + x)$$

and then dividing by Cx^n,

$$10 = 9x(1 + x).$$

This equation has two roots: $x = -\frac{1}{2} \pm \frac{7}{6}$. Since $|x| < 1$, we conclude $x = -\frac{1}{2} + \frac{7}{6} = \frac{2}{3}$. Finally, since $p_0 + p_1 + \cdots = 1$, it follows that $C = \frac{1}{3}$, so

$$p_n = \frac{1}{3} \left(\frac{2}{3}\right)^n, \quad n \geq 0.$$

c) $$E(L) = \sum_{n=0}^{\infty} n p_n = 2$$

and thus by Little's law, $E(S) = \frac{E(L)}{\lambda} = \frac{1}{5}$ hours.

d) Let r_n be the rate per hour that a nurse picks up n patients. Then, since nurses pick up patients at an exponential rate,

$$r_0 = \mu p_0 = 9 \cdot \frac{1}{3} = 3, \quad r_1 = \mu p_1 = 9 \cdot \frac{2}{9} = 2, \quad r_2 = \mu - r_0 - r_1 = 4.$$

4. Jobs arrive at a rate of $\lambda = \frac{1}{6}$ jobs per minute. The process time of a job is exponential with a mean of 3 minutes with probability $\frac{2}{5}$, and it is the sum of two independent exponentials, each with a mean of 3 minutes, with probability $\frac{3}{5}$. Hence $E(B) = \frac{2}{5} \cdot 3 + \frac{3}{5} \cdot 6 = \frac{24}{5}$ minutes, and $\rho = \lambda E(B) = \frac{4}{5}$.

a) The mean residual service time $E(R) = \frac{5}{8} \cdot 3 + \frac{3}{8} \cdot 6 = \frac{33}{8}$ minutes. Hence

$$E(W) = \frac{\rho E(R)}{1-\rho} = \frac{\frac{4}{5} \cdot \frac{33}{8}}{\frac{1}{5}} = \frac{33}{2},$$

so $E(S) = E(W) + E(B) = \frac{33}{2} + \frac{24}{5} = 21.3$ minutes.

b) Let ρ_i denote occupation rate due to jobs with i exponential operations, so $\rho_1 = \frac{2}{5} \cdot \frac{1}{6} \cdot 3 = \frac{1}{5}$ and $\rho_2 = \frac{3}{5}$. Then

$$E(W_1) = \frac{\rho E(R)}{1-\rho_1} = \frac{\frac{33}{10}}{\frac{4}{5}} = \frac{1}{4}\frac{33}{2} = \frac{33}{8}$$

and

$$E(W_2) = \frac{\rho E(R)}{(1-\rho_1)(1-\rho)} = \frac{5}{4}\frac{33}{2} = 5\frac{33}{8} = \frac{165}{8} \text{ minutes.}$$

So $E(S_1) = E(W_1) + 3 = 7\frac{1}{8}$ minutes and $E(S_2) = E(W_2) + 6 = 26\frac{5}{8}$ minutes.

c) Let T denote the exponential turn-on time, with a mean of 10 minutes. Then

$$E(W) = E(L^q)E(B) + \rho E(R) + (1-\rho)E(T).$$

Note that, if on arrival, the machine is idle or being turned on, the mean (residual) turn-on time is $E(T)$ (due to the memoryless property of exponentials). Hence, with Little's law, $E(L^q) = \lambda E(W)$,

$$E(W) = \frac{\rho E(R)}{1-\rho} + E(T) = \frac{33}{2} + 10 = 26.5$$

and $E(S) = E(W) + E(B) = 31.3$ minutes.

d) For jobs with 1 exponential operation, we obtain

$$E(W_1) = E(L_1^q)3 + \rho E(R) + (1-\rho)E(T)$$

and thus, with $E(L_1^q) = \lambda_1 E(W_1)$,

$$E(W_1) = \frac{\rho E(R)}{1-\rho_1} + \frac{1-\rho}{1-\rho_1}E(T) = \frac{33}{8} + \frac{1}{4} \cdot 10 = 6\frac{5}{8}$$

so $E(S_1) = E(W_1) + 3 = 9\frac{5}{8}$ minutes. Similarly

$$E(W_2) = E(L_1^q)3 + E(L_2^q)6 + \rho E(R) + (1-\rho)E(T) + \rho_1 E(W_2),$$

so with $E(L_2^q) = \lambda_2 E(W_2)$,

$$E(W_2) = \frac{E(L_1^q)3 + \rho E(R) + (1-\rho)E(T)}{1-\rho} = \frac{E(W_1)}{1-\rho} = \frac{265}{8} = 33\frac{1}{8}$$

so $E(S_2) = E(W_2) + 6 = 39\frac{1}{8}$ minutes.

Credits:

1a	b	c	d	2a	b	c	d	3a	b	c	d	4a	b	c	d
2	3	3	2	3	2	3	2	2	3	2	3	2	3	2	3