Final exam for the course 191551200 "Scientific Computing"
June 20, 2016, 08:45–10:15

It is not allowed to use any electronic equipment or books.
All the answers must be clearly motivated.

Part T1 (30 points)

T1.1 (8 p) Assume you have computed an LU factorization of a real $n \times n$ symmetric matrix A in Matlab as follows:

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[L, U, P, Q] = lu (A);
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Here it holds $PAQ = LU$, where P and Q are permutation matrices. How could you see, based on the matrices P, Q, L and U, whether A is positive definite?

If you like, you can also give here Matlab commands to check positive definiteness of A (in these commands, it is not allowed to compute any factorizations, matrix inverses or solve linear systems). Explain your answer, giving a theoretical motivation where necessary.

T1.2 (4 p) Give a definition of a stable numerical algorithm.

T1.3 (8 p) Give definitions of a normal matrix and the Schur decomposition of a matrix[1].

T1.4 (10 p) For an $n \times n$ matrix A holds $AA^* = A^*A$. What can be said about the Schur decomposition of A?

Part T2 (30 points)

T2.1 (6 p) Give a definition of a Krylov subspace method for solving a linear system $Ax = b$ (with given real $n \times n$ matrix A and vector b). After that formulate the Galerkin condition for a Krylov subspace method (if this condition is satisfied then the method is called the FOM, Fully Orthogonal Method).

T2.2 (10 p) Based on your answer to the previous question[2], derive one step of the FOM. In other words, explain in detail how, for a given x_0, compute the next iterand x_1. You are not allowed here to refer to the Arnoldi process (and it is easier not to do so).

T2.3 (8 p) We solve a linear system $Ax = b$ with given real $n \times n$ matrix $A \neq A^T$ and vector b. The Cholesky factorization of the Hermitian part of A exists and can be easily computed. Specify a two-sided preconditioned linear system such that the preconditioned matrix $\tilde{A}$ has its Hermitian part equal to the identity matrix. Prove this property of $\tilde{A}$.

See the other side

[1] If you forgot the definition of the Schur decomposition then ask me. You will get the definition from me and lose 4 points. You will need the definition for the next question.

[2] Ask me, if could not answer it.

T2.4 (6 p) We solve an initial-value problem for a system of ordinary differential equations

$$y'(t) = -Ay(t) + g(t), \qquad y(0) = y^0,$$

where a real $n \times n$ matrix A, vector y^0 and a real-valued vector function $g(t)$ are given. Let $\tau > 0$ be a time step size and let y^k be a numerical solution at time $t = k\tau$, $y^k \approx y(k\tau)$. Formulate the implicit midpoint rule scheme for this problem, involving vectors y^k and y^{k+1}. Rewrite it as a linear system where the unknown vector x is the update vector: $x = y^{k+1} - y^k$.

Part T3 (30 points)

T3.1 (10 p) For given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{k\times k}$, $C \in \mathbb{R}^{n\times k}$, we solve the Sylvester equation $AX - XB = C$. The matrix A is singular and skew-symmetric, i.e., $A^T = -A$. The matrix B is symmetric. Does the Sylvester equation have a unique solution for any C and for any symmetric B? Motivate your answer.

T3.2 (10 p) Let f be a sufficiently smooth function $f : \mathbb{R}^n \to \mathbb{R}$. Formulate the secant method for solving the gradient equation $\nabla f = 0$ and the quasi-Newton condition on the approximate Hessian matrix B_k. Hint: If you have difficulty with this question, formulate first the secant method for solving a scalar equation $F(x) = 0$ for $F : \mathbb{R} \to \mathbb{R}$.

T3.3 (10 p) For a nonlinear mapping F acting from $\mathbb{R}^n$ to $\mathbb{R}^n$, we solve the equation $F(x) = 0$.
(a) Formulate, for a given preconditioner matrix M, a preconditioned problem $\tilde{F}(x) = 0$, specifying $\tilde{F}(x)$.
(b) For the preconditioned problem, consider the fixed point iteration with the mapping $\tilde{K}(x) = x - \tilde{F}(x)$. How would you choose M for the first several iterations? How would you choose M when the first several iterations are done?

The grade is determined as $G = (10 + p)/10$, where p is the total number of points earned.