Practice Exam for Stochastic Processes

The following practice exam consists of four problems worth a total of 36 points. Motivate all your anwsers and be as thorough as possible. When a derivation is required, you must provide the full derivation.
Good luck!

Problem 1 [9 points]

A machine functions for an amount of time having distribution F with mean μ and variance σ^2. When the machine is out of order, it is immediately replaced by another one which has the same lifetime distribution F, ect. Let $m(t)$ be the mean number of replacements of the machine up to time t and let $Y(t)$ be the excess or residual lifetime of the machine working at time t.

In a) and b), assume that F is an arbitrary distribution.

a) [3pt] Prove that

$$m(t) = F(t) + \int_0^t m(t-x)dF(x), \quad t \geq 0.$$

b) [2pt] A new machine costs c_1 euro and the price of the energy and maintenance is c_2 euro per unit time. Determine the costs incurred per unit time in order to keep the system running.

In c) and d), assume that F is an Erlang-2 distribution:

$$F(x) = 1 - e^{-\frac{2x}{\mu}} - \frac{2x}{\mu}e^{-\frac{2x}{\mu}}, \quad x \geq 0.$$

c) [2pt] Determine $\lim_{t\to\infty} \mathbb{E}\left[Y(t)\right]$.

d) [2pt] Give an approximation for $m(t)$ when t is large.

Problem 2 [7 points]

Consider the Markov chain $\{Z_n\}_{n\geq 0}$ with state space $E = \{0, 1, \ldots, m\}$, $Z_0 = z_0$ and transition probabilities

$$p_{ij} = \begin{cases} 1, & i = j = 0 \text{ or } i = j = m \\ \binom{m}{j}\left(\frac{i}{m}\right)^j\left(1-\frac{i}{m}\right)^{m-j}, & \text{otherwise.} \end{cases}$$

a) [2pt] Show that $\{Z_n\}_{n\geq 0}$ is a martingale.

b) [3pt] Compute the probability of absorption by state 0.

c) [2pt] Use the Martingale Convergence Theorem to show that $\{Z_n\}_{n\geq 0}$ converges with probability one to a random variable Z. Use b) to write down the distribution of Z.

Problem 3 [8 points]

Let $\{S_n\}_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$, i.e. $S_n = \sum_{i=1}^{n} X_i$ and $S_0 = 0$ with $\{X_i\}_{i\geq 0}$ a sequence of i.i.d. random variables such that

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

For a fixed $a \in \mathbb{N}$ define

$$T = \min\{n \in \mathbb{N} : |S_n| = a\}.$$

a) [2pt] Show that $M_n = S_n^2 - n$ is a martingale.

b) [2pt] Show that $\mathbb{E}[T] = a^2$.

Consider now the case of a biased random walk, namely

$$\mathbb{P}(X_i = 1) = p > \frac{1}{2} \quad \text{and} \quad \mathbb{P}(X_i = -1) = q = 1 - p.$$

Define $Y_n = e^{bS_n - cn}$ for constants b and c and define

$$T_1 = \min\{n \in \mathbb{N} : S_n = 1\}.$$

c) [2pt] Derive a necessary relation between the constants b and c for which Y_n is a martingale.

d) [2pt] Find the moment generating function $\mathbb{E}\left[e^{-cT_1}\right]$ for $c > 0$.

Problem 4 [12 points]

Let $B(t)$ be a standard Brownian motion and define the Ornstein-Uhlenbeck process as

$$Z(t) = e^{-t}B(e^{2t}), \quad -\infty < t < \infty.$$

A stochastic process $\{X(t), t \geq 0\}$ is said to be stationary if for all $n, s, t_1, \ldots, t_n$ the random vectors $X(t_1), \ldots, X(t_n)$ and $X(t_1+s), \ldots, X(t_n+s)$ have the same joint distribution.

a) [4pt] Show that a necessary and sufficient condition for a Gaussian process to be stationary is that $\mathrm{Cov}(X(s), X(t))$ depends only on $t-s$, $s \leq t$ and $\mathbb{E}[X(t)] = c$ for some $c < \infty$.

b) [3pt] Let χ be a standard normal random variable independent from $Z(t)$. Show that

$$Z(t+s) = e^{-s}Z(t) + \chi\sqrt{1 - e^{-2s}}.$$

Hint: first show that

$$Z(t+s) = e^{-(s+t)}B(e^{2t}) + e^{-(s+t)}\left(B(e^{2(s+t)}) - B(e^{2t})\right).$$

c) [3pt] Obtain the covariance for the Ornstein-Uhlenbeck process.

d) [2pt] Show that $Z(t)$ is a stationary Gaussian process.