191531750 Stochastic Processes
Exam. Date: 03-02-2017, 13:45-16:15

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 4 problems. The total number of points is 36. Good luck!

Problem 1. Let $X_1, X_2, \ldots$ be i.i.d. non-negative continuous random variables with $\mathbb{E}(X) = \mu$ and $Var(X) = \sigma^2$. Define

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i,\ n = 1, 2, \ldots.$$

a) [3pt] Let $N(t)$ be the number of indices n, for which $0 < S_n \leq t$. Give an approximation for the expectation of $N(t)$ when t is large.

b) [5pt] Define $\phi(\theta) = \mathbb{E}(e^{\theta X_1})$. Assume that for some $\theta > 0$, $\phi(\theta)$ is finite and satisfies $\phi(\theta) \geq 1$. Fix an arbitrary $a > 0$ and define the stopping time

$$T = \min\{n : S_n > a\}.$$

Prove the Wald's identity:

$$\mathbb{E}\left[\phi(\theta)^{-T} e^{\theta S_T}\right] = 1.$$

Problem 2. You have a bag of coins, some of which are biased, meaning that if you flip a biased coin it will show HEAD $= 1$ with probability $a = 3/5$ and TAIL $= 0$ with probability $1 - a = 2/5$. The unbiased coins in the bag will give HEAD and TAIL with equal probability $b = 1/2$. You draw a single coin from the bag and want to test if this coin is biased by repeatedly flipping it. Let $\theta \in \{a, b\}$ denote the (unknown) bias of the coin you have drawn and let X_i denote the outcome of the i-th flip in trial.

Let $Z_0 = 1$ and

$$Z_n = \frac{p_{n,b}(X_1, \ldots, X_n)}{p_{n,a}(X_1, \ldots, X_n)},$$

for $n = 1, 2, \ldots$, where

$$p_{n,t}(X_1, \ldots, X_n) = t^{\sum_{i=1}^{n} X_i}(1 - t)^{n - \sum_{i=1}^{n} X_i}.$$

a) [3 pt] Show that $\{Z_n, n = 0, 1, \ldots\}$ is a martingale with respect to $\{X_n, n = 0, 1, \ldots\}$ if and only if $\theta = a$.

b) [2 pt] Show that if $\theta = a$, then $\{Z_n, n = 0, 1, \ldots\}$ converges with probability one.

c) [4 pt] Suppose that the coin that you have drawn is biased, i.e. $\theta = a$. Compute the distribution of $\lim_{n\to\infty} Z_n$. (Hint: *First consider* $\lim_{n\to\infty} \frac{1}{n} \log Z_n$.)

Problem 3. Let $\{X(t), t \geq 0\}$ be a standard Brownian motion and define $\{Y(t), t \geq 0\}$ as

$$Y(t) = X(t) - tX(1),$$

for $t \in [0, 1]$. Moreover, define $\{Z(t), t \geq 0\}$ as

$$Z(t) = (1-t)X\left(\frac{t}{1-t}\right),$$

for $t \in [0, 1)$ and $Z(1) = 0$.

a) [1 pt] Give the definition of a standard Brownian motion.

b) [1 pt] Give the definition of a Gaussian process.

c) [2 pt] Prove that Brownian motion is a Gaussian process.

d) [2 pt] Give $\mathbb{E}[X(t)]$ and $\mathrm{Cov}(X(s), X(t))$, for any $s \geq 0$, $t \geq 0$.

e) [4 pt] Prove that $\{Y(t), t \geq 0\}$ and $\{Z(t), t \geq 0\}$ are identical processes.

Problem 4. Let $\{X(t), t \geq 0\}$ be Brownian motion with drift $\mu < 0$ and variance $\sigma^2 = 1$.

a) [3pt] Prove that $\{Y(t), t \geq 0\}$ with

$$Y(t) = \exp\left(-2\mu X(t)\right),$$

is a martingale. (Hint: *If* $Z \sim \mathcal{N}(\mu, \sigma^2)$ *then* $\mathbb{E}[\exp(Z)] = \exp(\mu + \sigma^2/2)]$.)

b) [2pt] Prove that P_a, the probability that $\{X(t), t \geq 0\}$ hits level $a > 0$ before it hits level $b < 0$, is

$$P_a = \frac{1 - \exp(-2\mu b)}{\exp(-2\mu a) - \exp(-2\mu b)}.$$

You may invoke the martingale stopping theorem without verifying that all required conditions hold.

c) [4pt] Use the result from **b)** to find an expression for $P\left(\max_{0\leq t<\infty} X(t) \geq a\right)$.

Total: 36 points