153175 Stochastic Processes
Exam. Date: 01-02-2008, 13:30-16:30

In all answers: motivate your answer. When derivation is required, you must provide the derivation. It is not allowed to state that answer is by analogy with results in the book of Ross. This exam consists of 4 exercises. Good luck!

1. User's impatience is an important issue in telecommunications. Consider the following system. Packages arrive according to a Poisson process with parameter λ. The transmission time of a package has a probability distribution F with mean μ. The user chooses to interrupt a transmission if it takes longer than a given time T. If a transmission of a package is interrupted, then the entire package has to be retransmitted. A retransmission commences immediately after an interruption occurs. Each retransmission obeys the same rules as an initial transmission, that is, the time needed for a retransmission is independent of the past and has the distribution F. Furthermore, if a retransmission takes longer than T then it is to be interrupted by the user, and a new retransmission is to be started immediately. The retransmission attempts continue until the package is transmitted successfully. All packages that arrive when the channel is busy are lost.

   a) Give an expression for the long run throughput, that is the long run number of successfully transmitted packets per time unit.

   b) Compute the long-run throughput for a system where the user never interrupts a transmission.

   c) Looking at the results in a) and b), do you think it is possible that the user's impatience leads to a higher throughput? Justify your answer.

2. Suppose that $X_1, X_2, \ldots$ are independent random variables with

   $$\mathbb{P}[X_i = 1] = p, \quad \mathbb{P}[X_i = -1] = q, \quad i = 1, 2, \ldots,$$

   where $0 < p = 1 - q < 1$ and $p \neq q$. Conciser a random walk started at a positive integer value a:

   $$S_0 = a, \quad S_n = a + \sum_{i=1}^{n} X_i, \; n = 1, 2, \ldots.$$

   a) Find $\alpha \neq 1$ such that $Z_n = \alpha^{S_n}$ is a martingale. (*Hint: depending on the way you solve it, you might need the elementary identity* $(p+q)^2 = 1$).

   b) Take a positive integer b and let $T = \inf\{n \geq 1 : S_n = 0 \text{ or } S_n = a + b\}$ be the time epoch when S_n reaches 0 or $a + b$ for the first time. Argue that T is a stopping time for Z_n. Use the Martingale Stopping Theorem to find the probability $\mathbb{P}(S_T = b)$. Why is the Stopping Theorem applicable?

   c) In b), what happens if $q > p$ and $b \to \infty$? Could you predict this result using your knowledge on random walks?

d) Suppose that T_1 and T_2 are stopping times for some sequence $Y_1, Y_2, \ldots$. Is $\min\{T_1, T_2\}$ a stopping time? Justify your answer.

3. Let $S_n = \sum_{i=1}^{n} X_i$ be a random walk.

a) Show that a simple random walk in dimension 1 is recurrent, i.e., that it returns infinitely often to state 0.

b) Now consider the random walk in dimension 1 with

$$P\{X_i = j\} = \begin{cases} q, & j = -1 \\ \alpha_j, & j \geq 1 \end{cases} \qquad q + \sum_{j=1}^{\infty} \alpha_j = 1.$$

Let λ_i, $i > 0$, denote the probability that a ladder height equals i, that is $\lambda_i = P\{$first positive value of S_n equals $i\}$. Show that λ_i satisfies

$$\lambda_i = \alpha_i + q(\lambda_{i+1} + \lambda_1 \lambda_i), \qquad i > 0.$$

4. Let $B(t)$ be standard Brownian motion, and define the Ornstein-Uhlenbeck process

$$Z(t) = \mathrm{e}^{-t} B(\mathrm{e}^{2t}), \qquad -\infty < t < \infty$$

A stochastic process $\{X(t), t \geq 0\}$ is said to be a stationary process if for all $n, s, t_1, \ldots, t_n$ the random vectors $X(t_1), \ldots, X(t_n)$ and $X(t_1 + s), \ldots, X(t_n + s)$ have the same joint distribution.

a) Show that a necessary and sufficient condition for a Gaussian process to be stationary is that $Cov(X(s), X(t))$ depends only on $t - s$, $s \leq t$, and $E[X(t)] = c$.

b) Let χ be an independent standard normal. Show that

$$Z(t + s) = \mathrm{e}^{-s} Z(t) + \chi \sqrt{1 - \mathrm{e}^{-2s}}$$

Hint: first show that

$$Z(t + s) = \mathrm{e}^{-(s+t)} B(\mathrm{e}^{2t}) + \mathrm{e}^{-(s+t)} (B(\mathrm{e}^{2(s+t)}) - B(\mathrm{e}^{2t}))$$

c) Obtain the covariance for the Ornstein-Uhlenbeck process.

d) Show that $Z(t)$ is a stationary Gaussian process.

1(a)	1(b)	1(c)	2(a)	2(b)	2(c)	2(d)	3(a)	3(b)	4(a)	4(b)	4(c)	4(d)	Total
4	3	2	3	3	1	2	3	3	4	3	3	2	36