153175 Stochastic Processes
Final Exam. Date: 07.04.2008 Time: 13:30-16:30

This exam consists of 4 exercises. It is required to motivate all your answers. Good luck!

1. A production process on a factory yields waste that is temporarily stored on the factory site. The amounts of waste that are produced in successive weeks are independent and identically distributed with mean μ and variance σ^2. Opportunities to remove the waste from the factory site occur at the end of each week. The following control rule is used. If the amount of waste present is larger than D, then all the waste present is removed; otherwise, nothing is removed. There is a fixed cost of $K > 0$ for removing the waste and a variable cost of $v > 0$ for each unit of waste in excess of the amount D.

   a) [2pt] Define a regenerative process and identify its regeneration epochs.

   b) [3pt] Assume that the waste has just been removed, and let $m(s)$ be the average number of weeks such that the amount of waste does not exceed $s > 0$. Determine a long-run average cost per unit time.

   c) [3pt] Assuming that D is sufficiently large compared to μ, give an approximation to the long-run average cost.

   d) [2pt] Let $\{N(t), t \geq 0\}$ be a renewal process and $Y(t)$ be an excess time at epoch $t > 0$. Use the renewal reward argument to determine the long-run average excess:

$$\lim_{t\to\infty} \frac{\int_0^t Y(s)\, ds}{t}.$$

2. Define random variables recursively by $X_0 = 1$, and for $n \geq 1$, X_n is chosen uniformly at random on $(0, X_{n-1})$. Let $U_1, U_2, \ldots$ be independent uniform random variables on $(0, 1)$.

   a) [1pt] Verify that $X_n = U_n U_{n-1} \cdots U_1$.

   b) [3pt] Show that $Z_n = 2^n X_n$ is a martingale.

   c) [2pt] Use the fact that $\log X_n = \log U_1 + \cdots + \log U_n$ to show that

$$(1/n)\log(X_n) \to -1 \text{ with probability 1 as } n \to \infty.$$

   d) [2pt] Argue that $\lim_{n\to\infty} Z_n$ exists with probability 1. Use (c) to show that, with probability 1, $\lim_{n\to\infty} Z_n = 0$.

3. Let $\{S_n,\ n \geq 0\}$ be a random walk.

   a) [1 pt] Give the definition of a random walk.

   b) [1 pt] Give the duality principle.

   c) [1 pt] Give the definition of an ascending ladder variable.

   d) [2 pt] Give the probability that exactly n ascending ladder variables occur.

   e) [4 pt] Prove the following result: Suppose $X_1, X_2, \dots$ are independent and identically distributed random variables with $\mathbb{E}[X] > 0$. If

$$N = \min\{n : X_1 + \cdots + X_n > 0\},$$

   then

$$\mathbb{E}[N] < \infty.$$

4. Let $\{X(t),\ t \geq 0\}$ Brownian motion with drift coefficient μ. Let T_a denote the time $\{X(t),\ t \geq 0\}$ hits a.

   a) [1 pt] Give the definition of Brownian motion with drift coefficient μ.

   b) [4 pt] For $\mu > 0$, let $g(x) = Var(T_x)$, and derive a differential equation for $g(x)$, $x > 0$.

   c) [2 pt] What is the relationship between $g(x)$, $g(y)$, and $g(x + y)$.

   d) [2 pt] Solve the differential equation obtained in b) for $g(x)$.

   Hint: use the conditional variance formula $Var(x) = \mathbb{E}[Var(X|Y)] + Var(\mathbb{E}[X|Y])$

   Total: 36pt+4pt=40pt.