191531750 Stochastic Processes
Exam. Date: 09-04-2013, 13:45-16:45

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 7 problems. The total number of points is 36.

Good luck!

1. Consider a renewal process with a distribution $F(\cdot)$ and expectation μ of the times between successive renewals. Let $m(t)$ denote the renewal function of this process.

   a) [2pt] Argue that $m(t)$ satisfies the following equation:

   $$m(t) = F(t) + \int_0^t m(t-x)\mathrm{d}F(x), \quad t \geq 0.$$

   b) [3pt] Assume that $F(\cdot)$ is not arithmetic. Let δ_t be the age at $t > 0$. Derive $\lim_{t\to\infty} P(\delta_t > x)$.

2. A production process on a factory yields waste that is temporarily stored on the factory site. The amounts of waste that are produced in successive weeks are independent and identically distributed random variables $X_1, X_2, \ldots$, with mean μ and variance σ^2. Opportunities to remove the waste from the factory site occur at the end of each week. The following control rule is used. If the amount of waste present is larger than D, then all the waste present is removed; otherwise, nothing is removed. There is a fixed cost of $K > 0$ for removing the waste and a variable cost of $v > 0$ for each unit of waste in excess of the amount D.

   a) [2pt] Let $m(s)$ be the renewal function of the renewal process where the times between successive renewals are $X_1, X_2, \ldots$. Express the average number of weeks between two successive removals of the waste in terms of $m(s)$.

   b) [2pt] Assuming that D is sufficiently large compared to μ, give an approximation for the total costs paid by the factory each time when a waste is removed.

3. [2pt] Consider two random variables:

   $$T_1 = \{\min n \geq 3 : S_n = S_{n-3}\}, \quad T_2 = T_1 - 3.$$

   Is T_1 a Markov time? Is T_2 a Markov time?

4. Let $\{X_i\}$ be a sequence of random variables such that $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = -1) = 1 - p = q < \frac{1}{2}$. Define the partial sums

$$S_0 = 0; \quad S_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1.$$

Define also

$$T_1 = \min\{n \in \mathbb{N} : S_n = 1\}.$$

a) [2pt] Find $E(T_1)$. You use the fact that $E(T_1) < \infty$.

b) [2pt] Define $Y_n = e^{bS_n - cn}$ for constants b and c. Derive a necessary relation between the constants b and c in order that Y_n is a martingale.

c) [3pt] Find the moment generating function $\mathbb{E}(e^{-cT_1})$ for $c > 0$.

5. Consider the bounded martingale $\{M_n\}_{n=0}^{\infty}$. Let $X_n = \sum_{k=1}^{n} \frac{1}{k}(M_k - M_{k-1})$.

a) [2pt] Show that $\{X_n\}_{n=1}^{\infty}$ is a martingale.

b) [2pt] Prove that $\{X_n\}$ converges with probability one.

c) [2pt] Prove that $\{X_n\}$ converges in the mean.

6. Let $\{X(t), t \geq 0\}$ be a standard Brownian motion and define the Brownian bridge

$$Z(t) = X(t) - tX(1) \qquad \text{for} \quad 0 \leq t \leq 1.$$

a) [2pt] Is $\{-X(t), t \geq 0\}$ a Brownian motion? Is $\{X(2t) - X(t), t \geq 0\}$ a Brownian motion?

b) [3pt] Find the covariance function of the Brownian bridge for all $0 \leq s \leq t \leq 1$.

c) [2pt] Show that $\{B(t), t \geq 0\}$ is a Brownian motion for $B(t)$ defined by $B(t) = (1 + t)Z(\frac{t}{1+t})$.

7. Let $\{X(t), t \geq 0\}$ be a standard Brownian motion.

a) [3pt] Prove Kolmogorov's inequality for Brownian motion:

$$P\left(\sup_{0 \leq u \leq t} |X(u)| > \epsilon\right) \leq t/\epsilon^2, \quad \epsilon > 0.$$

b) [2pt] Let $Y(t) = \sup_{0 \leq u \leq t} X(u) - X(t)$. Derive the distribution of $Y(t)$.

Total: 36 points