191531750 Stochastic Processes
Exam. Date: 08-04-2014, 13:45-16:45

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 4 problems. The total number of points is 36.
Good luck!

1. A lazy professor has a ceiling fixture in his office that contains two light bulbs. To replace a bulb, the professor must fetch a ladder, and being lazy, when a single bulb fails, he waits until the second bulb fails before replacing them both at the same time. Assume that the lengths of life of the bulbs are independent and exponentially distributed with parameter 2 (we choose one year as a time unit, so the average life time of a bulb is 1/2 year).

   a) [2pt] Give the definition of a renewal process.

   b) [3pt] What fraction of time, in the long run, is our professor's office half lit?

   c) [3pt] Provide an approximation for the average number of bulbs that the professor have used in n years, such that the difference between the exact average value and your approximation goes to zero as $n \to \infty$. Compute the approximation for the average number of bulbs used in 15 years.

   d) [2pt] Explain why a Poisson process with intensity λ is a stationary process.

2. The winnings per unit stake on game n is X_n, where $X_1, X_2, \ldots$ are independent identically distributed random variables with $\mathbb{P}(X_n = 1) = p > 1/2$, $\mathbb{P}(X_n = -1) = 1 - p = q$. At game n, a gambler bets an amount $c_n Z_{n-1}$, where Z_{n-1} is the gambler's fortune after $n-1$ games, and $c_n \in [0, 1]$ is the fraction of fortune that he is willing to bet. The gambler plays a fixed number N of games. The goal is to optimize the 'interest rate', $\mathbb{E}[\log(Z_N/Z_0)]$, where Z_0 is the gambler's initial fortune.

   a) [3pt] Show that for any strategy $(c_1, c_2, \ldots, c_N)$, the process $\log(Z_n/Z_0) - n\alpha$ is a supermartingale, where

   $$\alpha = p\log(p) + q\log(q) + \log(2).$$

   Hint: Use the fact that $p\log(1+x) + q\log(1-x)$ is maximized when $x = p - q$.

   b) [3pt] Use the Martingale Stopping Theorem for supermartingales to show that for any strategy $(c_1, c_2, \ldots, c_N)$ holds $\mathbb{E}[\log(Z_N/Z_0)] \leq N\alpha$. Prove that the Stopping Theorem is applicable.

   c) [2pt] The gambler chooses $c_n = p - q$ for all n. Argue that $\log(Z_n/Z_0) - n\alpha$ is a martingale, and the interest rate $\mathbb{E}[\log(Z_N/Z_0)]$ equals $N\alpha$. Use b) to argue that $c_n = p - q$ is the gambler's optimal strategy.

3. Consider the Markov chain $\{Z_n\}_{n\geq 0}$ with state space $E = \{0, 1, \dots, m\}$, $Z_0 = z_0$, and transition probabilities

$$p_{ij} = \begin{cases} 1, & i = j = 0 \text{ or } i = j = m \\ \binom{m}{j}\left(\frac{i}{m}\right)^j\left(1 - \frac{i}{m}\right)^{m-j}, & \text{otherwise.} \end{cases}$$

a) [2pt] Show that $\{Z_n\}_{n\geq 0}$ is a martingale.

b) [3pt] Compute the probability of absorption by state 0.

c) [3pt] Use the Martingale Convergence Theorem to show that $\{Z_n\}_{n\geq 0}$ converges with probability one to a random variable Z. Use b) to write down the distribution of Z.

4. Let $B(t)$ be standard Brownian motion, and define the Ornstein-Uhlenbeck process

$$Z(t) = \mathrm{e}^{-t} B(\mathrm{e}^{2t}), \qquad -\infty < t < \infty$$

A stochastic process $\{X(t), t \geq 0\}$ is said to be a stationary process if for all $n, s, t_1, \dots, t_n$ the random vectors $X(t_1), \dots, X(t_n)$ and $X(t_1+s), \dots, X(t_n+s)$ have the same joint distribution.

a) [3 pt] Show that a necessary and sufficient condition for a Gaussian process to be stationary is that $Cov(X(s), X(t))$ depends only on $t-s$, $s \leq t$, and $E[X(t)] = c$.

b) [2pt] Let χ be an independent standard normal. Show that

$$Z(t+s) = \mathrm{e}^{-s} Z(t) + \chi\sqrt{1 - \mathrm{e}^{-2s}}$$

Hint: first show that

$$Z(t+s) = \mathrm{e}^{-(s+t)} B(\mathrm{e}^{2t}) + \mathrm{e}^{-(s+t)}(B(\mathrm{e}^{2(s+t)}) - B(\mathrm{e}^{2t}))$$

c) [3pt] Obtain the covariance for the Ornstein-Uhlenbeck process.

d) [2pt] Show that $Z(t)$ is a stationary Gaussian process.

Total: 36 points