153175 Stochastic Processes
Exam. Date: 29-01-2010, 13:45-16:45

In all answers: motivate your answer. When derivation is required, you must provide the derivation. This exam consists of 4 problems. The total number of points is 36.

Good luck!

1. a) [2pt] Give the definition of the renewal reward process and provide a formula for computing the long-run average reward.

   b) [2pt] Consider an ergodic discrete-time Markov chain with state space $\mathcal{S} = \{1, 2, \ldots, N\}$. Let π_i be a stationary probability that the chain is in state $i = 1, \ldots, N$. Use the renewal reward process to show that for any $i = 1, \ldots, N$ holds $\pi_i = 1/m_{ii}$, where m_{ii} is the average time between two successive visits to state i.

   c) [2pt] Let $\{N(t), t \geq 0\}$ be a renewal process, where the time intervals between renewals have mean μ and variance σ^2, and $Y(t)$ be an excess time at epoch $t > 0$. Use the renewal reward argument to prove that

   $$\lim_{t\to\infty} \frac{\int_0^t Y(s)\,ds}{t} = \frac{\sigma^2 + \mu^2}{2\mu}.$$

   In d) and e), consider an inventory of one product. The demands in successive weeks are independent and identically distributed with mean μ and variance σ^2. If the stock is insufficient to satisfy the demand then the unsatisfied demand is backlogged, and it will be satisfied once the stock is replenished. The possibilities to replenish occur at the end of every week, and the replenishment orders arrive instantaneously. The following control rule is used. Initially the stock is S. The replenishment order is placed if the stock is empty. The replenishment is sufficient to satisfy the backlog demand and to bring the stock back to level S.

   d) [2pt] Write the number of weeks between two successive replenishments using a suitable renewal process. [*Hint:* Think which random variables play the role of the inter-arrival times and which event causes the placement of a replenishment order.]

   e) [2pt] Assuming that S is large compared to μ, what is the approximate size of one replenishment order?

2. Let $(S_n)_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$, i.e. $S_n = \sum_{i=1}^n X_i$ (where $S_0 = 0$), with $(X_i)_{i\geq 0}$ a sequence of i.i.d. random variables such that

   $$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2.$$

   For fixed $a, b \in \mathbb{N}$ define $T = \min\{n \in \mathbb{N} : S_n = -a \text{ or } S_n = b\}$.

   a) [2pt] Show that for any λ the process $Z_n = (\cos(\lambda))^{-n} \cos\left(\lambda\left(S_n - \frac{b-a}{2}\right)\right)$ is a martingale. [*Hint:* Use the formula $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$) and the properties: $\sin(-x) = -\sin(x)$, $\cos(-x) = \cos(x)$.]

b) [2pt] Show that for any n holds $\mathbb{E}\left[\cos\left(\lambda\left(S_n - \frac{b-a}{2}\right)\right)\right] = (\cos(\lambda))^n \cos\left(\lambda\left(\frac{b-a}{2}\right)\right)$

c) [2pt] Fix an arbitrary N and let $T_N = \min\{N, T\}$. Use the martingale stopping theorem to show that

$$\mathbb{E}\left[(\cos(\lambda))^{-T_N} \cos\left(\lambda\left(S_{T_N} - \frac{b-a}{2}\right)\right)\right] = \cos\left(\lambda\left(\frac{b-a}{2}\right)\right).$$

Why is the stopping theorem applicable?

d) [2pt] If $0 < \lambda < \pi/(a+b)$, then it can be shown that the stopping theorem applies to the stopping time T. Use this fact to determine $\mathbb{E}[(\cos(\lambda))^{-T}]$, $0 < \lambda < \pi/(a+b)$.

3. Let $X_1, X_2, \ldots$ iid random variables taking values in $\mathbb{R}$, and $S_0 = 0$, $S_n = \sum_{i=1}^n X_i$, $n = 1, 2, \ldots$ a random walk in $\mathbb{R}$.

a) [1pt] Let K_i, $i = 1, 2, \ldots$ be a sequence of ascending ladder epochs. Give the definition of the first ascending ladder variable and the first ascending ladder epoch K_1.

b) [1 pt] Consider the n cyclic arrangements of $(X_1, \ldots, X_n)$, numbered as follows:

number	arrangement
0	$(X_1, \ldots, X_n)$
1	$(X_2, \ldots, X_n, X_1)$
$\vdots$	$\vdots$
$n-1$	$(X_n, X_1, \ldots, X_{n-1})$

Let $S_j^{(k)}$ be the sum of the first j elements in arrangement k. Show that

$$S_j^{(k)} = \begin{cases} S_{k+j} - S_k & j = 0, \ldots, n-k \\ S_n - S_k + S_{j-n+k} & j = n-k+1, \ldots, n \end{cases}$$

c) [2 pt] Prove the following theorem: For arbitrary fixed $n \geq 1$, let r be the number of arrangements for which n is a ladder epoch. Then for each of these arrangements, n is the r-th ladder epoch. Furthermore, $r \geq 1$ if and only if $S_n \geq 0$.

[Hint: draw a picture of arrangement 0 that includes the ladder epochs and continue the picture until time $n + K_1$]

d) [1 pt] For the cyclic arrangements of $(X_1, \ldots, X_n)$ and each fixed $r \geq 1$, let

$$J_s = \begin{cases} 1 & \text{if } n \text{ is the } r \text{ th ladder epoch in arrangement } s \\ 0 & \text{otherwise} \end{cases} \qquad s = 0, \ldots, n-1$$

Show that the J_s are identically distributed.

e) [1 pt] Show that $J_0 + \cdots + J_{n-1}$ can take only two possible values: 0 and r.

f) [1 pt] Let $\alpha_m = P(K_1 = m)$, $m = 1, 2, \ldots$, with generating function $\mathcal{A}(z) = E[z^{K_1}] = \sum_{m=1}^{\infty} \alpha_m z^m$. Show that

$$E[J_0] = \alpha_n^{(r)} = rP(J_0 + \cdots + J_{n-1} = r)/n,$$

where $\alpha_n^{(r)}$ is the r-fold convolution of α_n with itself.

g) [2 pt] Prove the following theorem on the distribution of the ladder epochs, where ln denotes the natural logarithm:

$$\ln \frac{1}{1 - \mathcal{A}(z)} = \sum_{n=1}^{\infty} z^n P(S_n > 0)/n$$

4. Let $\{B(t),\ t \geq 0\}$ be standard Brownian motion.

a) [1 pt] Give the definition of standard Brownian motion.

b) [2 pt] Give the definition of geometric Brownian motion, and argue why geometric Brownian motion is useful in modeling a process in which percentage changes are independent and identically distributed.

c) [2 pt] Show that

$$Y(t) = \exp\{cB(t) - c^2 t/2\}$$

is a martingale.

d) [1 pt] Suppose one has the option of purchasing at fixed price K, at a time t in the future, one unit of a stock with price that follows

$$X(t) = X(0)\exp(\mu t + \sigma B(t)), \quad t \geq 0.$$

Let the interest rate be r, then the discounted stock price is $e^{-rt}X(t)$, where multiplication is with e^{-rt} since 1 euro at time t has the same value as e^{-rt} at time 0.

Give the value for μ for which the discounted stock price is a martingale.

e) [3 pt] Black-Scholes formula: The value of the option at time 0 is $E[e^{-rt}(X(t) - K)^+]$. Show that

$$E[e^{-rt}(X(t) - K)^+] = X(0)\Phi(\sigma\sqrt{t} - \alpha) - e^{-rt}K\Phi(-\alpha)$$

where $\alpha = [\ln(K/X(0)) - \mu t]/\sigma\sqrt{t}$, $\Phi(x) = P(\mathcal{N}(0,1) \leq x)$, and $\mathcal{N}(a,b)$ is the normal random variable with mean a and variance b.

Total: 36 points