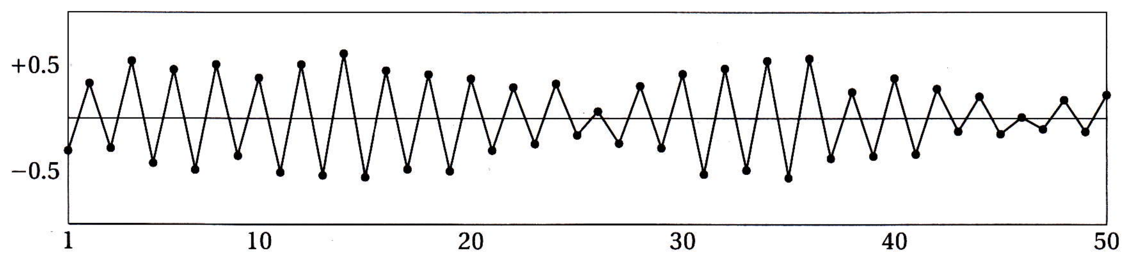
Time Series Analysis (& SI)—191571090

(The first 1.5 hours it is CLOSED BOOK. The second 1.5 hours is OPEN BOOK.)

Date: 30-10-2015
Place: THERM
Time: 08:45–11:45



1. **CLOSED BOOK:** Three questions:

   (a) Have a look at the above realization $x_1, \ldots, x_{50}$ of the AR scheme $X_t = aX_{t-1} + \epsilon_t$. What is your guess for $\mathbb{E}(\epsilon_t)$ and a and σ_ϵ?

   (b) What is definition of *cross covariance function* of two stochastic processes Y_t and X_t.

   (c) In the notes it is claimed that the standard biased estimator

   $$\hat{r}_N(k) := \frac{1}{N} \sum_{t=0}^{N-|k|-1} (X_{t+|k|} - \hat{m}_N)(X_t - \hat{m}_N)$$

   has two clear advantages over the unbiased estimator $\frac{1}{N-|k|}\sum_{t=0}^{N-|k|-1}(X_{t+|k|} - \hat{m}_N)(X_t - \hat{m}_N)$. Mention at least one of the two advantages.

2. **CLOSED BOOK:** Consider the MA scheme $X_t = \epsilon_t + b\epsilon_{t-1}$ and suppose that $|b| < 1$ and that $\mathbb{E}\epsilon_t = 0$.

   (a) Determine $r(k)$ in terms of b

   (b) Determine the one-step ahead predictor $\hat{X}_{t+1|t}$.

   (c) Compute the mean square prediction error $\mathbb{E}(\hat{X}_{t+1|t} - X_{t+1})^2$

   (d) The standard predictor $\hat{X}_{t+1|t}$ has access to the present and entire past of X_t. We could also choose to estimate X_{t+1} as a function of the present X_t alone: determine the $c \in \mathbb{R}$ that minimizes $\mathbb{E}(X_{t+1} - cX_t)^2$

   (e) For the optimal c of the previous part compute $\mathbb{E}(X_{t+1} - cX_t)^2$. Is it less than σ_ϵ^2?

   (f) Your answers for (a–e) where derived under the assumption that $|b| < 1$. Which of the answers of (a–e) still hold if $|b| > 1$?

3. **OPEN BOOK:** Suppose X_t is iid normally distributed white noise with variance $\sigma^2 > 0$. Suppose we know that the mean is zero. Given are samples $x_1, \dots, x_N$.

   (a) Show that the maximum likelihood estimator of σ^2 is $\hat{\sigma}^2_{\text{ML}} = \frac{1}{N}(\sum_{i=1}^N x_i^2)$.

   (b) Is $\hat{\sigma}^2_{\text{ML}}$ unbiased?

   (c) Is $\hat{\sigma}^2_{\text{ML}}$ efficient? [you may use that $\mathbb{E}(Z^4) = 3\sigma_Z^4$ for normally distributed Z]

4. **OPEN BOOK:** In Exercise 7.7 of the notes it is claimed if X_t is a zero mean AR(1) process, $X_t = aX_{t-1} + \epsilon_t$, then the estimate $\hat{\theta}$ of the model $X_t = \theta X_{t-1} + \tilde{\epsilon}_t$ based on N samples of X_t satisfies

$$\operatorname{var}(\hat{\theta}) \approx (1 - a^2)/N.$$

   (a) Explain in words why $\operatorname{var}(\hat{\theta})$ does not depend σ_ϵ.

   (b) The formula implies that $a = 0.9$ is "easier" to estimate than $a = 0.5$.
   Explain in words that this makes sense using that $r_X(k) = \dfrac{\sigma_\epsilon^2}{1-a^2} a^{|k|}$.

5. **OPEN BOOK:** Suppose X_t is zero mean normally distributed WSS process with covariance function

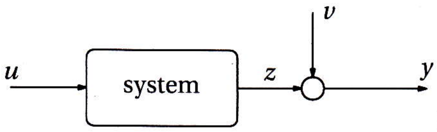
$$r_X(k) = \begin{cases} 4 & \text{if } k = 0 \\ 2 & \text{if } k = \pm 1 \\ 0 & \text{elsewhere} \end{cases}$$

   How many samples N of X_t should we collect before the estimate $\hat{r}_N(0)$ has a standard deviation of less than 0.01?

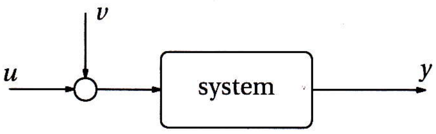
6. **OPEN BOOK:** Let U_t and V_t be two zero mean WSS processes and suppose the two processes are uncorrelated. Assume further that the system $\mathscr{H}$ is LTI. In the notes it is proved that frequency response $\hat{h}(\omega)$ of the system equals

$$\hat{h}(\omega) = \frac{\phi_{yu}(\omega)}{\phi_u(\omega)}$$

   for scheme (a) of the figure below. Does the result also hold for scheme (b) of the figure below? If so, show it. If not, derive a formula for $\hat{h}(\omega)$ in terms of $\phi_{yu}(\omega)$ and $\phi_u(\omega)$.



(a)



(b)

problem:	1	2	3	4	5	6
points:	3+2+2	2+2+2+2+2+3	2+2+3	2+2	2	3

Exam grade is $1 + 9p/p_{\text{max}}$.