Exam "Discrete Optimization"

Monday, January 12, 2015, 13:30 – 16:30

- Use of calculators, mobile phones, and other electronic devices is not allowed!
- The exam consists of six problems. Please start a new page for every problem.
- The total number of points of the regular assignments is 45. You get 5 bonus points, and you can get 5 bonus points from Exercise 1b.

1. Independent Set

We consider the problem **IndependentSet**:

Instance: undirected graph $G = (V, E)$;

Solution: subset $U \subseteq V$ such that, for all $u, v \in U$, we have $\{u, v\} \notin E$ (the set U is called an *independent set*);

Goal: maximize $|U|$.

Let $d_v = |\{u \in V \mid \{u, v\} \in E\}|$ be the degree of a vertex $v \in V$ in G, and let $\Delta = \max\{d_v \mid v \in V\}$ be the maximum degree of G.

In order to approximate the problem, we consider the greedy method: we start with $U = \emptyset$. Then we consider the nodes of V one by one. If we consider $v \in V$ and $\{v\} \cup U$ is an independent set, then we add it to U. Otherwise, we skip v.

(a) *(6 points)* Prove that the greedy method achieves an approximation ratio of $\Delta + 1$ for IndependentSet. This means that greedy outputs an independent set U such that $|U| \geq \frac{1}{\Delta+1} \cdot |U^\star|$, where $U^\star$ denotes an independent set of maximum cardinality.

(b) *(5 bonus points)* Prove that greedy achieves an approximation ratio of Δ for IndependentSet.

If you solve this part correctly, you also get the points of Part (a).

2. Shortest Paths

(6 points) Devise an algorithm with a running-time of $O(n+m)$ for the following optimization problem:

Instance: directed graph $G=(V,E)$ with $V=\{v_1,\dots,v_n\}$ and $|E|=m$ and edge costs $c:E\to\mathbb{R}$ (negative edge costs are allowed). The graph satisfies the following property: if $(v_i,v_j)\in E$, then $i<j$.

Goal: Compute shortest path distances from v_1 to all other nodes in G.

A proof of correctness and of the running-time is not required.

3. Spanning Trees

We consider the problem **SecondMST** of computing the second-lightest spanning tree in a graph:

Instance: undirected graph $G=(V,E)$, edge weights $w:E\to\mathbb{N}$;

Solution: a spanning tree S (called the second-lightest tree) of G with the following properties:

- There exists a tree $T^\star \neq S$ with $w(T^\star)\leq w(S)$.
- We have $w(T)\geq w(S)$ for all spanning trees $T\neq T^\star$ of G.

Remark: In the following, the observation that $|T\triangle T'|$ is even for all spanning trees T and T' is useful. The observation holds since $|T|=|T'|=n-1$.

(a) *(5 points)* Let $T^\star$ be a minimum-weight spanning tree, and let Y be an arbitrary spanning tree with $|Y\triangle T^\star|\geq 4$. Then there exists a spanning tree Z with the following two properties:

  (i) $|Z\triangle T^\star| = |Y\triangle T^\star| - 2$.

  (ii) $w(T^\star)\leq w(Z)\leq w(Y)$.

(b) *(3 points)* Let $T^\star$ be a minimum-weight spanning tree. Prove the following: There exists a second-lightest tree S with $|S\triangle T^\star|=2$.

  Remark: You can of course use Part (a) even if you did not prove it.

(c) *(4 points)* Devise a polynomial-time algorithm for SecondMST. Justify the correctness of your algorithm and estimate its running-time.

  Remark: You can of course use Part (b) even if you did not prove it.

4. NP-Completeness

The **bounded-degree spanning tree problem**, denoted by **BoundedMST**, is the following decision problem:

Instance: undirected graph $G = (V, E)$;

Question: is there a spanning tree T of G such that every node $u \in V$ is incident to at most three edges in T?

(7 points) Prove that BoundedMST is NP-complete.

Hint: You can reduce from **HamiltonPath**, which is the following problem and known to be NP-complete:

Instance: undirected graph $G = (V, E)$;

Question: is there a simple path P in G that visits all vertices of G (called a Hamilton path)?

Note that a Hamilton path is a spanning tree of maximum degree two.

5. Minimum Cost Flows

(6 points) Let $G = (V, E)$ be a directed graph with edge capacities $w : E \to \mathbb{N}$ and edge costs $c : E \to \mathbb{N}$ and balances $b : V \to \mathbb{Z}$.

Prove that the following statements are equivalent for all feasible flows f:

(I) The flow f is the unique minimum cost flow.

(II) For every directed cycle C in G_f, we have $c(C) > 0$.

6. Questions

Are the following statements true or false? Give a short justification of your answer.

(a) *(2 points)* Consider the minimum cost flow problem. If all balances, capacities, and costs are integral, then all optimal flows have integral costs.

(b) *(2 points)* Since Knapsack can be solved in pseudo-polynomial time, there is no polynomial-time many-one reduction from 3SAT to Knapsack.

(c) *(2 points)* $\mathsf{PerfectMatch} = \{G \mid G \text{ contains a perfect matching}\} \in \mathsf{NP}$.

(d) *(2 points)* Let $G = (V, E)$ be a directed graph and let $s \in V$ be arbitrary. Let $c : E \to \mathbb{R}$ be edge costs such that G does not contain any negative cycle. Then there exists a node $v \in V$ such that the shortest path from s to v consists of a single edge.