Re-Exam "Discrete Optimization"

Wednesday, January 18, 2017, 13:00 – 16:00

- Answers have to be in English.
- Use of calculators, mobile phones, and other electronic devices is not allowed.
- The exam consists of six problems. Please start a new page for every problem.
- The total number of points is 50.

1. Matroids

Let $G = (V, E)$ be an undirected, connected graph. Let $n = |V|$ be the number of vertices of G. We assume that $|E| \geq n$. A 1-tree of G is a connected spanning subgraph L that has exactly n edges. (The name 1-tree comes from the fact that L consists of a spanning tree plus one additional edge.)

Let $(E, \mathcal{M})$ be the following independent set system:

- If $F \subseteq E$ forms a 1-tree, then $F \in \mathcal{M}$.
- If $F' \subseteq F$ and F is a 1-tree, then $F' \in \mathcal{M}$.

(8 points) Prove that $(E, \mathcal{M})$ is a matroid. To do this, you only have to prove that (M3) holds: If $A, B \in \mathcal{M}$ and $|A| < |B|$, then there exists some edge $e \in B \setminus A$ with $A \cup \{e\} \in \mathcal{M}$.

2. NP-Completeness

The hitting set problem HittingSet is the following optimization problem:

Instance: a finite set X (called the "universe"); subsets $S_1, \ldots, S_n \subseteq X$.

Solution: a subset $H \subseteq X$ that satisfies $H \cap S_i \neq \emptyset$ (this means that H "hits" all the subsets $S_1, \ldots, S_n$ – hence H is called a "hitting set").

Goal: minimize $|H|$.

The decision version of HittingSet is the following problem: Given an instance of HittingSet and a number $k \in \mathbb{N}$, does there exist a hitting set H with $|H| \leq k$?

(a) *(7 points)* Prove that HittingSet is NP-hard. You do not have to prove that the decision version of HittingSet is in NP.

Hint: VertexCover is the following NP-hard problem:

Instance: undirected graph $G = (V, E)$.

Solution: $U \subseteq V$ such that each edge in E has at least on endpoint in U. (Then U is called a "vertex cover" of G.)

Goal: minimize $|U|$.

The decision version of VertexCover is the following problem: Given an instance of VertexCover and an $\ell \in \mathbb{N}$, does there exist a vertex cover U of G with $|U| \leq \ell$?

(b) *(7 points)* Consider the following algorithm for HittingSet, which we call NAIVE:

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H = ∅
while there is some i with S_i ∩ H = ∅ do
    H = H ∪ S_i
end while
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Let $c_{\max} = \max\{|S_i| \mid 1 \leq i \leq n\}$ be the cardinality of the largest set in the given instance.

Prove that NAIVE yields a $c_{\max}$-approximation for HittingSet. This means that NAIVE computes a hitting set U such $|U| \leq c_{\max} \cdot |U^\star|$, where $|U^\star|$ is a hitting set of minimum cardinality.

3. Pseudo-Polynomial and Approximation Algorithms

Pack is the following optimization problem:

Instance: numbers $a_1, \dots, a_n \in \mathbb{N}$, a number $b \in \mathbb{N}$

Solution: a subset $I \subseteq \{1, \dots, n\}$ with $\sigma(I) = \sum_{i \in I} a_i \leq b$

Goal: maximize $\sigma(I)$.

(a) *(5 points)* Devise an algorithm that solves Pack in time polynomial in b and n. It suffices if your algorithm outputs $\sigma(I^\star)$, where $I^\star$ denotes an optimal solution. Your algorithm does not need to output the set $I^\star$.

You do not have to prove the correctness of your algorithm.

(b) *(5 points)* Devise a polynomial-time algorithm that computes a set I such that $\sigma(I) \geq \frac{1}{2} \cdot \sigma(I^\star)$, where $I^\star$ denotes an optimal solution.

This means that your algorithm should be a 2-approximation for Pack.

4. Min-Cost Flows

We consider flow networks $G = (V, E)$ with balances $b = (b_v)_{v \in V}$ and edge costs $c = (c_e)_{e \in E}$, but without edge capacities. This means that f is a flow if f assigns a non-negative flow value to each directed edge and satisfies the balance constraints. We assume that there exists a feasible flow f.

(6 points) Prove that the following two statements are equivalent for all such flow networks:

(I) There exists a minimum-cost flow in G.

(II) The flow network G does not contain a directed cycle of negative costs.

5. Minimum Spanning Trees

(4 points) Prove the following statement.

For all undirected, connected graphs $G = (V, E)$ with edge weights w, the following holds: Let t be the smallest number such that $G_t = (V, E_t)$ is connected, where

$$E_t = \{e \in E \mid w_e \leq t\}.$$

Then G_t contains a minimum spanning tree of G.

6. Miscellaneous Questions

Are the following statements true or false? Justify your answer. This justification can be a short proof, a reference to a theorem of the lecture, a counterexample, ...

(a) *(2 points)* For all directed graphs $G = (V, E)$ with edge lengths d and $s, t \in V$, the following holds:

There exists a shortest s-t path in G if and only if

- there exists an s-t path in G and
- G does not contain a cycle of negative length.

(b) *(2 points)* Let $\mathsf{PerfectMatch} = \{G \mid G \text{ contains a perfect matching}\}$. If there is a polynomial-time many-one reduction from $\mathsf{3SAT}$ to $\mathsf{PerfectMatch}$, then $\mathsf{NP} = \mathsf{P}$.

(c) *(2 points)* For all decision problems A and B with $B \subseteq A$, the following holds: If $A \in \mathsf{P}$, then $B \in \mathsf{P}$.

(Here, $B \subseteq A$ means that the set of "yes" instances of B is a subset of the "yes" instances of A.)

(d) *(2 points)* Let $G = (V, E)$ be a connected graph consisting of at least three vertices, and let T be a spanning tree of G. Then, for all $X \subseteq V$ with $\emptyset \neq X \neq V$, there is exactly one edge in T crossing the cut $(X, \overline{X})$.