Mastermath and LNMB Course: Discrete Optimization

Exam 30 January 2012

Utrecht University, Buys Ballot Laboratorium, 15:00–18:00

The examination lasts 3 hours. Grading will be done before February 13, 2012. Students interested in checking their results can make an appointment by e-mail (g.schaefer@cwi.nl).

The examination consists of five problems. The maximum number of points to be gained on the different parts are displayed in the following table:

1(a)–(j)	2	3(a)	3(b)	4(a)	4(b)	5	$\sum$
20 (2 each)	15	5	5	15	10	15	**85**

The grade for the exam is obtained by dividing the total number of points by **8.5**. This implies that **47** points are needed to pass.

During the examination only the Lecture Notes of the course without any additional leaflets are allowed to be on your desk and all electronic equipment must be switched off.

Please be short, clear and precise in your answers. If you use results from the Lecture Notes, please provide the respective references.

Good Luck!

Problem 1 (20 points (2 points each)). State for each of the claims below whether it is **true** of **false**. Note: You need not justify or prove your answers here.

(a) $17n\log n = \Theta(n^2)$.

(b) Given a spanning tree T of a graph, the number of vertices that have odd degree in T is even.

(c) The symmetric difference $M_1 \triangle M_2$ of two arbitrary matchings M_1 and M_2 exclusively consists of even length cycles and isolated nodes.

(d) If all capacities in the max-flow problem are integers, then there exists a maximum flow that is integral.

(e) Let T be a minimum spanning tree of a graph $G = (V,E)$ with edge cost $c : E \to \mathbb{R}$. By removing an edge $e \in T$ from T, we obtain two trees whose node sets induce a cut $(X,\bar{X})$. Every edge e' that crosses $(X,\bar{X})$ satisfies $c(e') \geq c(e)$.

(f) A pseudoflow that satisfies all capacity constraints is a flow.

(g) If $\Pi_1 \in NP$ and for every problem $\Pi_2 \in NP$, $\Pi_1 \preceq \Pi_2$, then Π_1 is NP-complete.

(h) If $\Pi_1 \in NP$ and $\Pi_2 \preceq \Pi_1$ for some NP-complete problem Π_2, then Π_1 is NP-complete.

(i) The TSP problem in which all edge distances are either 1 or 2 is NP-complete.

(j) Suppose ALG is an α-approximation algorithm for an optimization problem Π whose approximation ratio is tight. Then for every $\varepsilon > 0$ there is no $(\alpha - \varepsilon)$-approximation algorithm for Π (unless $P = NP$).

Problem 2 (15 points). Consider the *maximum weight indegree-bounded subgraph problem*: We are given a directed graph $G = (V,E)$ with edge weights $w : E \to \mathbb{R}^+$ and degree bounds $b : V \to \mathbb{N}$. The goal is to determine a subset $E' \subseteq E$ of maximum total weight $w(E') = \sum_{e\in E'} w(e)$ such that every node $u \in V$ has indegree at most $b(u)$.

Prove that the greedy algorithm for matroids solves this problem.

Problem 3 ($5+5$ points). The *Hitchcock problem* is as follows: We are given a set of m sources $M = \{1,\dots,m\}$ and a set of n terminals $N = \{1,\dots,n\}$. Every source $i \in M$ has a supply of $s(i) \in \mathbb{N}$ units and every terminal $j \in N$ has a demand of $d(j) \in \mathbb{N}$ units. We assume that $\sum_{i\in M} s(i) = \sum_{j\in N} d(j)$. The cost to send one unit from source $i \in M$ to terminal $j \in N$ is given by $c(i,j) \in \mathbb{R}^+$. An *allocation* specifies for each pair $(i,j) \in M \times N$ the amount $x(i,j)$ that is sent from i to j. An allocation is *feasible* if it satisfies the supply of every source $i \in M$ and the demand of every terminal $j \in N$. The goal is to compute a feasible allocation x of minimum total cost $\sum_{(i,j)\in M\times N} c(i,j)x(i,j)$.

(a) Formulate the *Hitchcock problem* as an integer linear program and derive the respective LP relaxation.

(b) Show that the set of feasible solutions of this LP is an integral polytope.

Problem 4 ($15+10$ points). In the *longest s,t-path problem* we are given a directed graph $G=(V,E)$ with edge weights $w : E \to \mathbb{R}^+$, a source node $s \in V$ and a target node $t \in V$. The goal is to compute a simple s,t-path P of maximum total weight $w(P) = \sum_{e \in P} w(e)$.

(a) The decision variant of the problem is to determine whether there exists a simple s,t-path of total weight at least K, where K is a given parameter. Show that this problem is NP-complete. (Hint: Use that the *Hamiltonian path problem* is NP-complete: Given an undirected graph G, determine whether G contains a Hamiltonian path.)

(b) Show that the *longest s,t-path problem* in acyclic graphs can be solved in time $O(n+m)$, where n and m refer to the number of nodes and edges of G, respectively.

Problem 5 (15 points). Consider the *maximum weight acyclic subgraph problem*: We are given a directed graph $G=(V,E)$ with edge weights $w : E \to \mathbb{R}^+$. The goal is to determine a subset $E' \subseteq E$ of maximum total weight $w(E') = \sum_{e \in E'} w(e)$ such that the subgraph $G' = (V, E')$ induced by E' is acyclic.

Derive a 2-approximation algorithm for this problem and show that its approximation ratio is tight. (Hint: Assign a unique number $r(u)$ to every node $u \in V$ of G. An edge $(u,v) \in E$ is a *forward edge* if $r(u) < r(v)$. Prove that the set of all forward edges induces an acyclic subgraph of G.)