Exam: Linear Structures 1.
Applied Mathematics, 2017-1A: Structures and Models
October 31 2017; 8:45 - 11:45

This exam consists of 9 problems. A (graphical) calculator is not needed and can be used only to check your answer. The following is IMPORTANT and will be taken into account for grading: i) Define all variables and explain notations that you introduce in your solution. ii) Clearly explain each step of your solution, in words or by a clear derivation.

1. [10pt] Consider a set $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. The addition and scalar multiplication are defined as follows:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 2a_2 + 2b_2), \quad c(a_1, a_2) = (ca_1, ca_2), \; c \in \mathbb{R}.$$

Prove that V is not a vector space by showing that at least one of the following properties fail:

(VS 7) for all $u, v \in V$, $c \in F$ holds $c(u + v) = cu + cv$,

(VS 8) for all $v \in V$, $c, d \in F$ holds $(c + d)v = cv + dv$.

2. [10pt] Prove that $\{2x^2 + 3x + 1, 3x^2 + 4x + 2, -x^2 + x + 2\}$ is a basis for $P_2(\mathbb{R})$.

3. Let V, W be linear spaces with finite dimensions. Let $T : V \to W$ be a linear transformation. Assume that T is one-to-one.

(a) [10pt] Let β be a basis for V. Prove that the set $T(\beta)$ is linearly independent.

(b) [5pt] Prove that $\dim(W) \geq \dim(V)$.

4. $T : \mathbb{R}^4 \to M_{2\times2}(\mathbb{R})$ is given by

$$T((a, b, c, d)) = \begin{pmatrix} a & b + c \\ 3(b + c) & -2d \end{pmatrix}.$$

(a) [5pt] Determine $[T]_\beta^\gamma$ where β is the standard basis for $\mathbb{R}^4$ and γ is the standard basis for $M_{2\times2}(\mathbb{R})$.

(b) [5pt] Verify the statement $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$ for $v = (2, 1, -1, 2)$.

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5. [10pt] $Ax = b$ is a linear system. Prove that A is invertible if and only if $s = A^{-1}b$ is the unique solution to this linear system.

6. Denote by $\underline{a}_j$, $j = 1, 2, 3, 4, 5$, column j of a 3×5 matrix A. Suppose that

$$\underline{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \underline{a}_2 = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \underline{a}_4 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

The reduced row echelon form of the augmented matrix $(A\ b)$ of the system $Ax = b$ is as follows:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{pmatrix}.$$

(a) [5pt] Determine the solution set of this linear system.

(b) [5pt] Explain how to obtain the solution of the homogeneous system $Ax = \mathbf{0}$ and write down this solution.

(b) [5pt] Determine $\underline{a}_3$, $\underline{a}_5$ and b.

7. [5pt] Give an example of a 3×3 matrix C with different non-zero rows, such that rank$(C) = 1$. Argue that C indeed has rank 1. Write the matrix C in the form $C = AB$, where A is 3×1 matrix and B is 1×3 matrix.

8 [10pt] Let $T : V \to V$ be a linear transformation. Prove that the set of vectors

$$S = \{v : T(v) = \lambda v, \lambda \in F\} \subseteq V$$

is a subspace of V.

9. [5pt] Find the eigenvalues of the matrix A:

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Total: 90 points

NB: grade=([number of points]+10)/10.