Course : **Mathematics β1, Bernoulli**

Date : October 27, 2017
Time : 13.45 – 16.45 hrs

Motivate all your answers.
The use of electronic devices is not allowed.

1. [3 pt] Solve the following initial value problem

$$\begin{cases} y' + 2xy = e^{-x^2}, \\ y(0) = 5. \end{cases}$$

2. (a) [3 pt] Define $w = \dfrac{2}{1+i}$. Give $|w|$ and $\arg(w)$.
Calculate w^8 by using De Moivre's Theorem.

   (b) [3 pt] Given the two sets $A = \{z \in \mathbb{C} \mid z^3 = 8i\}$ and $B = \{z \in \mathbb{C} \mid Re(z) \neq 0\}$. Determine all elements in $A \cap B$.

3. a) [3 pt] Find the general solution $y(x)$ which solves the second order differential equation

$$4y'' + 5y' + y = 10e^x.$$

Is $y(x)e^{-x}$ convergent if $x \to \infty$?

   b) [3 pt] Solve the given initial-value problem

$$\begin{cases} 4y'' + 5y' + y = 10e^x + 2x + 5, \\ y(0) = -1, \\ y'(0) = 0. \end{cases}$$

4. Given are the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ with the property:

$$\mathbf{u} \perp \mathbf{v}, \quad \mathbf{u} \perp \mathbf{w}, \quad \mathbf{v} \perp \mathbf{w}.$$

   (a) [3 pt] Give a proof for the statement $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = 0$.

   (b) [2 pt] Determine if the following statement is true or false. Motivate your answer.

$$|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}|\,|\mathbf{v}|\,|\mathbf{w}|$$

5. Given are the points $P(3,4,-5)$ and $Q(5,5,0)$ in $\mathbb{R}^3$.

(a) [1pt] Determine the vector equation of the line ℓ going through P and Q.

(b) [2 pt] Let W be the plane through the origin, P and Q. Determine an equation of W.

6. Calculate the following limits:

(a) [2 pt] $\lim_{x\to\infty}\left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right)$,

(b) [2 pt] $\lim_{x\to 0}\frac{1-\cos(x)}{3x^2}$.

7. The function $f:\mathbb{R}\to\mathbb{R}$ is given by

$$f(x)=\begin{cases}\sqrt{1-x^2} & \text{if } -1\le x\le 1\\ 0 & \text{if } x<-1 \text{ or } x>1.\end{cases}$$

(a) [2 pt] Show that f is continuous at 1.

(b) [2 pt] Is f differentiable at 1?

(c) [3 pt] Determine the extrema of f in the interval $[-1,1]$.

(d) [2 pt] Let g be a function on $\mathbb{R}$. The following statement is the formal definition of "g is continuous at $a\in\mathbb{R}$". Write the negation of this statement without using the $\neg$-operator

$$\forall\epsilon>0(\exists\delta>0(\forall x\in\mathbb{R}(|x-a|<\delta\to|g(x)-g(a)|<\epsilon))).$$

Total: 36 points