Mathematics A (Euclides) and B1 (Leibniz)

Solutions/correction standard for test Mathematics A + B1, Nov 4, 2013

1. (a) [1 pt] For example: p: "He wears a white T-shirt" and q: "The T-shirt he is wearing is not green". Then $p \to q$ is true, but $q \to p$ is false. Of course there are many more examples **[1 pt]**

(b) [3 pt] Membership table for $(A \cup B) - C$ and $(A - B) \cup (B - C)$:

A	B	C	$A \cup B$	$(A \cup B) - C$	$A - B$	$B - C$	$(A - B) \cup (B - C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	1	0	1	1	0	1	1
0	1	1	1	0	0	0	0
1	0	0	1	1	1	0	1
1	0	1	1	0	1	0	1
1	1	0	1	1	0	1	1
1	1	1	1	0	0	0	0

[1.5 pt]

Conclusion:
The fifth and last column are not identical, so the statement is false. **[0.5 pt]**
These columns only differ in the sixth row, so in order to find a counterexample, we must take sets A, B and C such there is an element in the intersection $A \cap C$ that is not in B. For example $A = C = \{1\}$ and $B = \varnothing$.
Then $(A \cup B) - C = \varnothing$, but $(A - B) \cup (B - C) = \{1\}$. **[1 pt]**

Incorrect table: -0.5 pt for each incorrect column.
If table is not correct but the way the conclusion is deduced from the table is: 0.5 pt.
A counterexample that is deduced from an incorrect table that is not a counterexample for the statement in the exercise: 0 pt (a counterexample must be checked)

2. (a) [2 pt] Let $m, n \in \mathbb{Z}$ be both odd.
Then there exist $k, l \in \mathbb{Z}$ such that $m = 2k + 1$ and $n = 2l + 1$. **[0.5 pt]**
Then $mn = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$. **[1 pt]**
So mn can be written as $mn = 2s + 1$, with $s \in \mathbb{Z}$, and therefore mn is odd. **[0.5 pt]**

(b) [4 pt] Basis step for $n = 1$: $1^3 + 2 \cdot 1 = 3$, so the statement is correct for $n = 1$ (take $\ell = 1$).

[0.5 pt]

Induction step:
Let $k \geq 1$ and suppose that: $k^3 + 2k$ is divisible by 3, so $k^3 + 2k = 3\ell$ for some $\ell \in \mathbb{Z}$ (Induction hypothesis: IH). **[1 pt]**

We must show that IH implies:
$(k + 1)^3 + 2(k + 1)$ is divisible by 3. **[1 pt]**

Well:

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$$
$$= k^3 + 2k + 3(k^2 + k + 1). \qquad \textbf{[0.5 pt]}$$

Now IH implies that this is equal to $3\ell + 3(k^2 + k + 1)$. **[0.5 pt]**
Rewriting this last expression yields: $3(\ell + k^2 + k + 1)$
So $(k+1)^3 + 2(k+1)$ is divisible by 3. **[0.5 pt]**
Now we obtain from the principle of mathematical induction that for all $n \in \mathbb{N}$: $n^3 + 2n$ is divisible by 3.
(From the proof it must be crystal clear whàt is supposed **[1 pt]** and whàt must be proved **[1 pt]**. In case of nonsense formulations like "Suppose it is correct FOR ALL n, so it also holds for $n+1$": at most 1 pt for the entire exercise)

3. (a) [1 pt] For each digit there are 10 possibilities. So by the rule of product: there are 10^{12} different strings.
(answer: **[0.5 pt]**, (some) argumentation: **[0.5 pt]**).

(b) [3 pt] The number of ones must be 6, 7, 8 of 9.
First we determine the number of strings with exactly 3 zeros and exactly 6 ones. Choose 3 digits for the zeros, this can be done in $\binom{12}{3}$ ways. For each choice for the zeros, there are $\binom{9}{6}$ possibilities to determine the digits for the 6 ones and for each choice for the zeros and the ones, there are 8^3 possibilities for the remaining three digits (which cannot be 0 or 1). **[1 pt]**
Therefore, by the rule of product, the number of strings with exactly 3 zeros and exactly 6 ones is: $\binom{12}{3} \cdot \binom{9}{6} \cdot 8^3$. **[0.5 pt]**
Similarly,the number of strings with exactly 3 zeros and exactly 7, 8 or 9 ones is: $\binom{12}{3} \cdot \binom{9}{7} \cdot 8^2$, $\binom{12}{3} \cdot \binom{9}{8} \cdot 8^1$, and $\binom{12}{3} \cdot \binom{9}{9} \cdot 8^0$ respectively. **[1 pt]**

Therefore the number of strings with 3 zeros and at least 6 ones is equal to:

$$\binom{12}{3} \cdot \binom{9}{6} \cdot 8^3 + \binom{12}{3} \cdot \binom{9}{7} \cdot 8^2 + \binom{12}{3} \cdot \binom{9}{8} \cdot 8 + \binom{12}{3}. \qquad \textbf{[0.5 pt]}$$

(Just the answer, without any argumentation: **[1.5 pt]**).
Note that the answer $\binom{12}{3} \cdot \binom{9}{6} \cdot 9^3$ (choose 6 digits for the ones and take for the remainig three digits any nonzero digit) is wrong.

4. (a) [1 pt] The function $f(x) = e^{\sin(x)}$ is not one-to-one:. An answer without motivation (even if it is the correct answer): 0 pt. Provide two numbers a and b with $a \neq b$ and $f(a) = f(b)$, for example $a = 0$ and $b = \pi$. **[1 pt]**

(b) [2 pt] The range of $\sin(x)$ is $[-1, 1]$ The function e^x maps this interval to the interval $\left[\frac{1}{e}, e\right]$. **[1 pt]**
In order to conclude this, you need that e^x is an increasing function. If this argument is used: **[1 pt]**.

Note that in fact also the continuity of e^x is required. Since continuity is not part of the curriculum, there is no deduction for not using this argument.

5. (a) [2 pt] $\overrightarrow{RQ} = \langle 0, 2\sqrt{3}, 6\rangle$. **[0.5 pt]**

$\left|\overrightarrow{RQ}\right| = \sqrt{0+12+36} = \sqrt{48} = 4\sqrt{3}$. **[1 pt]**

The unit vector in the direction of $\overrightarrow{RQ}$ is $\langle 0, \frac{1}{2}, \frac{1}{2}\sqrt{3}\rangle$. **[0.5 pt]**

Note: deduct 0.5 pt if the answer is $\overrightarrow{QR} = \langle 0, -\frac{1}{2}, -\frac{1}{2}\sqrt{3}\rangle$.

(b) [2 pt] Write PQ and PR as vectors:

$$\mathbf{u} = \overrightarrow{PQ} = \langle 2, \sqrt{3}, 3\rangle \quad \text{and} \quad \mathbf{v} = \overrightarrow{PR} = \langle 2, -\sqrt{3}, -3\rangle.$$ **[0.5 pt]**

Calculate the lengths and the dot product of $\mathbf{u}$ and $\mathbf{v}$:

$$\begin{aligned} |\mathbf{u}| &= \sqrt{4+3+9} = 4, \\ |\mathbf{v}| &= \sqrt{4+3+9} = 4, \\ \mathbf{u}\cdot\mathbf{v} &= 4-3-9 = -8. \end{aligned}$$ **[0.5 pt]**

If the $\varphi = \angle RPQ$, then

$$\cos\varphi = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{-8}{16} = -\frac{1}{2}.$$ **[.5pt]**

From this follows:

$$\varphi = \frac{2}{3}\pi.$$ **[.5pt]**

(c) [2 pt] Calculate the cross product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u}\times\mathbf{v} = \begin{bmatrix} 2 & \sqrt{3} & 3 \\ 2 & -\sqrt{3} & -3 \end{bmatrix} \begin{matrix} 2 & \sqrt{3} \\ 2 & -\sqrt{3} \end{matrix} = \langle 0, 12, -4\sqrt{3}\rangle.$$ **[1pt]**

The surface area of the triangle is

$$\tfrac{1}{2}\,|\mathbf{u}\times\mathbf{v}| = \tfrac{1}{2}\sqrt{192} = 4\sqrt{3}.$$ **[1pt]**

6. (a) [2 pt] Calculate z^2:

$$\begin{aligned} z^2 &= \left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right)^2 \\ &= \left(\sqrt{2+\sqrt{3}}\right)^2 + \left(i\sqrt{2-\sqrt{3}}\right)^2 + 2i\sqrt{2+\sqrt{3}}\sqrt{2-\sqrt{3}} \\ &= 2+\sqrt{3} - \left(2-\sqrt{3}\right) + 2i\sqrt{(2+\sqrt{3})(2-\sqrt{3})} \\ &= 2\sqrt{3} + 2i\sqrt{4-3} \\ &= 2\sqrt{3} + 2i, \end{aligned}$$

so $\mathrm{Re}\left(z^2\right) = 2\sqrt{3}$ and $\mathrm{Im}\left(z^2\right) = 2$. **[2 pt]**

Deduct .5 pt for every error.

(b) [1 pt] Use the result from (a):

$$\begin{aligned} \left|z^2\right| &= \sqrt{\left(\operatorname{Re} z^2\right)^2 + \left(\operatorname{Im} z^2\right)^2} \\ &= \sqrt{\left(2\sqrt{3}\right)^2 + 2^2} \\ &= \sqrt{12+4} = 4. \end{aligned}$$ **[0.5 pt]**

For the argument φ of z^2 two methods can be used:

$$\tan\varphi = \frac{\operatorname{Im} z^2}{\operatorname{Re} z^2} = \frac{2}{2\sqrt{3}} = \frac{1}{3}\sqrt{3},$$

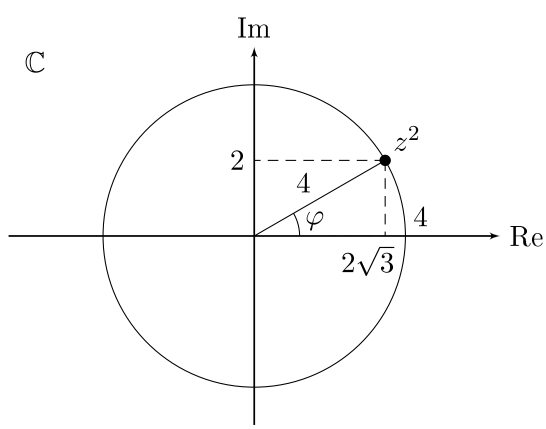
and z^2 is in the right-half plane, hence $\varphi = \dfrac{\pi}{6}$. **[0.5 pt]**

Note: no points are awarded if the argument 'z^2 is in the right-half plane' is missing, since $\tan$ is periodic with period π. The same holds if φ is calculated with $\arctan$:

$$\varphi = \arctan\left(\frac{\operatorname{Im} z^2}{\operatorname{Re} z^2}\right) = \dots$$

is wrong.

Alternatively, use a picture:



[0.5 pt]

(c) [1 pt] The number z^6 can be calculated in several ways. The most convenient is to use the result from (b):

$$\left|z^6\right| = \left|z^2\right|^3 = 4^3 = 64$$

and

$$\arg z^6 = 3\arg z^2 = 3\cdot\frac{\pi}{6} = \frac{\pi}{2}.$$ **[0.5 pt]**

Therefore

$$z^6 = 64\, e^{i\frac{\pi}{2}} = 64i,$$ **[0.5 pt]**

and consequently

$$\operatorname{Re}\left(z^6\right) = 0 \quad \text{and} \quad \operatorname{Im}\left(z^6\right) = 64.$$

Note: The real and imaginary part of z^6 need not to be presented in this way. The answer $z^6 = 64i$ will be accepted as correct, but $z^6 = 64\, e^{i\frac{\pi}{2}}$ is not.

Other methods (like Newton's binomium) may be used, but calculations might be tedious and are error prone. Nevertheless, if calculations are executed correctly and lead to the right answer ("$z^6 = 64i$" or "$\mathrm{Re}\left(z^6\right) = 0$ and $\mathrm{Im}\left(z^6\right) = 64$"), 1 point is awarded.

7. [2 pt] If $y(x) = \ln(1 + e^x)$, then by the chain rule:

$$y'(x) = \frac{1}{1+e^x} \cdot e^x. \qquad \textbf{[1 pt]}$$

Notice that $e^{y(x)} = 1 + e^x$, hence

$$\begin{aligned} y'(x) &= \frac{1}{1+e^x} \cdot e^x \\ &= \frac{1}{e^{y(x)}} \cdot e^x \\ &= e^{x-y(x)}. \end{aligned} \qquad \textbf{[1 pt]}$$

8. [3 pt] Rewrite the differential equation in the form $y' + a(x)y = b(x)$:

$$y' + \frac{4}{x}y = \frac{3}{x^2}. \qquad \textbf{[0.5 pt]}$$

The integrating factor is

$$v(x) = e^{\int a(x)\ \mathrm{d}x} = e^{\int \frac{4}{x}\ \mathrm{d}x} = e^{4\ln x} = x^4. \qquad \textbf{[0.5 pt]}$$

The general solution is

$$y(x) = \frac{1}{v(x)} \int v(x)b(x)\ \mathrm{d}x = \frac{1}{x^4} \int 3x^2\ \mathrm{d}x = \frac{x^3+c}{x^4}. \qquad \textbf{[1 pt]}$$

Use the initial condition to find c:

$$0 = y(1) = \frac{1^3+c}{1} \quad \Rightarrow \quad c = -1. \qquad \textbf{[0.5 pt]}$$

Finally, write down the solution:

$$y(x) = \frac{x^3-1}{x^4} \quad \text{or} \quad y(x) = \frac{1}{x} - \frac{1}{x^4}. \qquad \textbf{[0.5 pt]}$$

9. [4 pt] Step 1 (total: 1 pt): solve the homogeneous equation $y'' - 2y' + 2y = 0$.

The corresponding auxiliary or characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$. This equation has two imaginary roots:

$$\lambda = 1 + i \quad \text{and} \quad \lambda = 1 - i. \qquad \textbf{[0.5 pt]}$$

Therefore the general solution of the auxiliary equation is

$$y(x) = e^x(c_1 \cos(x) + c_2 \sin(x)). \qquad \textbf{[0.5 pt]}$$

Step 2 (total: 2 pt): find a particular solution.

We try a polynomial of degree 1, in other words: try

$$y_p(x) = ax + b \qquad \textbf{[0.5 pt]}$$

with unknown constants a and b. Notice that

$$y_p{}'(x) = a \quad \text{and} \quad y_p{}''(x) = 0, \qquad \textbf{[0.5 pt]}$$

hence

$$y_p{}'' - 2y_p{}' + 2y_p = 0 - 2a + 2(ax + b) = x + 2 \qquad \textbf{[0.5 pt]}$$

This leads to the following equations for a and b:

$$\begin{aligned} 2a &= 1, \\ 2b - 2a &= 2. \end{aligned} \qquad \textbf{[0.5 pt]}$$

Solve the equations: $a = \frac{1}{2}$ and $b = \frac{3}{2}$, so the particular solution is $y_p(x) = \frac{1}{2}x + \frac{3}{2}$.

Step 3 (total: 1 pt): determine the constants c_1 and c_2.

The general solution is

$$y(x) = e^x(c_1 \cos(x) + c_2 \sin(x)) + \tfrac{1}{2}x + \tfrac{3}{2}. \qquad (1)$$

From $y(0) = 0$ follows

$$0 = y(0) = 1 \cdot (c_1 \cdot 1 + c_2 \cdot 0) + \tfrac{3}{2},$$

hence $c_1 = -\frac{3}{2}$. **[0.5 pt]**

Differentiate (1):

$$y'(x) = e^x(-\tfrac{3}{2}\cos(x) + c_2 \sin(x)) + e^x(\tfrac{3}{2}\sin(x) + c_2\cos(x)) + \tfrac{1}{2}.$$

From $y'(0) = -1$ follows

$$-1 = y'(0) = 1 \cdot (-\tfrac{3}{2} \cdot 1 + c_2 \cdot 0) + 1 \cdot (\tfrac{3}{2} \cdot 0 + c_2 \cdot 1) + \tfrac{1}{2},$$

hence $c_2 = 0$. **[0.5 pt]**

The solution of the initial value problem is

$$y(x) = -\tfrac{3}{2}e^x \cos(x) + \tfrac{1}{2}x + \tfrac{3}{2}.$$