Solution to Exam Stochastic Differential Equations (Mastermath)
08-06-2015; 13:30 – 16:30.

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(5) 1. a. Let $X \in L^1$ be a random variable and let $\mathcal{F}$ and $\mathcal{G}$ be σ-algebras such that $\mathcal{F} \subseteq \mathcal{G}$. Show that $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{F}) = \mathbb{E}(X|\mathcal{F})$.

(5) b. Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of random variables. Assume there is a constant C such that for all integers $n \in \mathbb{N}$, $\mathbb{E}(|X_n|^2) \leq C$. Show that X is uniformly integrable.

(5) c. Let $(X_t)_{t\geq 0}$ be a nonnegative local martingale such that $\mathbb{E}(X_0) < \infty$. Show that X is a supermartingale.

(2) d. Let $0 < a < b$ and assume $\xi : \Omega \to \mathbb{R}$ is $\mathcal{F}_a$-measurable. Use the definition to show that $\mathbf{1}_{(a,b]}\xi$ is predictable.

(4) e. Use the previous exercise and an approximation argument to show that any left-continuous adapted process $Z : \mathbb{R} \times \Omega \to \mathbb{R}$ is predictable.

[Soln]

a. Let $Z = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{F})$. For every $F \in \mathcal{F}$ we also have that $F \in \mathcal{G}$ and thus by the definition of the conditional expectation we find that

$$\int_F Z \, d\mathbb{P} = \int_F \mathbb{E}(X|\mathcal{G}) \, d\mathbb{P} = \int_F X \, d\mathbb{P}.$$

Since Z is $\mathcal{F}$-measurable the required identity follows.

b. For each $r > 0$ we have

$$\int_{\{|X_n|>r\}} |X_n| \, d\mathbb{P} \leq r^{-1} \int_{\{|X_n|>r\}} |X_n|^2 \, d\mathbb{P} \leq r^{-1} \int_\Omega |X_n|^2 \, d\mathbb{P} \leq r^{-1} C.$$

Therefore,

$$\lim_{r\to\infty} \sup_{n\geq 1} \int_{\{|X_n|>r\}} |X_n| \, \mathbb{P} \leq \lim_{r\to\infty} r^{-1} C = 0.$$

c. See one of the exercises of chapter 3.

d. The definition of the predictability can be found in the lecture notes. There are at least two possible solutions:

1. Approximate ξ by $\mathcal{F}_a$-measurable simple functions ξ_n, then $\mathbf{1}_{(a,b]}\xi_n$ can be written as a linear combination of predictable rectangles and hence is predictable. Then also the pointwise limit $\mathbf{1}_{(a,b]}\xi$ is predictable.

2. Check that $\mathbf{1}_{(a,b]}\xi \in B$ is in the predictable σ-algebra. If $0 \notin B$ this is simple. If $0 \in B$, then some more rewriting is required.

e. Use d and approximation. See the lecture notes for details.

END

2. Let B be a standard Brownian motion and let $\mathcal{F}_t = \sigma(B_s : s \leq t)$.

(6) a. Let $\alpha \geq 0$. Using the properties of conditional expectations and the independent increments of Brownian motion show that $X_t = \cosh(\alpha|B_t|)\exp(-\alpha^2 t/2)$ is a martingale.

Hint: Recall that $\cosh(|x|) = \cosh(x) = (e^x + e^{-x})/2$. You may also use the identity: $\mathbb{E}(\exp(\xi)) = \exp(\sigma^2/2)$ for $\xi \sim N(0, \sigma^2)$.

Fix $A > 0$ and let $\tau = \inf\{t \geq 0 : |B_t| = A\}$.

(3) b. Prove that τ is a stopping time.

(3) c. Show that $\mathbb{E}(X_{t\wedge\tau}) = 1$ for all $t \geq 0$.

(5) d. Show that $\mathbb{E}(X_\tau) = 1$ and use this to find a formula for $\mathbb{E}(e^{-\lambda\tau})$ where $\lambda \geq 0$.

Hint: You may use the known fact that $\tau < \infty$ almost surely.

[Soln]

a. Note that for $s < t$ by linearity of the conditional expectation

$$\mathbb{E}(X_t|\mathcal{F}_s) = \frac{1}{2}\exp(-\alpha^2 t/2)\Big(\mathbb{E}(\exp(\alpha B_t)|\mathcal{F}_s) + \mathbb{E}(\exp(-\alpha B_t)|\mathcal{F}_s)\Big).$$

We calculate both conditional expectations.

$$\begin{aligned}\mathbb{E}(\exp(\alpha B_t)|\mathcal{F}_s) &= \mathbb{E}(\exp(\alpha(B_t - B_s))\exp(\alpha B_s)|\mathcal{F}_s)\\ &= \exp(\alpha B_s)\mathbb{E}(\exp(\alpha(B_t - B_s))|\mathcal{F}_s) \quad \text{(taking out what is known)}\\ &= \exp(\alpha B_s)\mathbb{E}(\exp(\alpha(B_t - B_s))) \quad \text{(independence)}\\ &= \exp(\alpha B_s)\exp(\alpha^2(t-s)/2) \quad (\alpha(B_t - B_s) \sim N(0, \alpha^2(t-s)))\end{aligned}$$

The other conditional conditional expectation can be calculated in the same way and is

$$\mathbb{E}(\exp(\alpha B_t)|\mathcal{F}_s) = \exp(-\alpha B_s)\exp(\alpha^2(t-s)/2).$$

Putting everything together we find

$$\mathbb{E}(X_t|\mathcal{F}_s) = \exp(-\alpha^2 t/2)\cosh(\alpha B_s)\exp(\alpha^2(t-s)/2) = X_s.$$

b. For every $t \geq 0$ we can write

$$\{\tau > t\} = \{\forall s \in [0,t]\ B_s < A\} = \{\forall s \in [0,t] \cap \mathbb{Q}\ B_s < A\} = \bigcap_{s \in [0,t] \cap \mathbb{Q}} \{B_s < A\} \in \mathcal{F}_t.$$

where in the last step we used the adaptedness of B and the fact that the *countable union* of sets in $\mathcal{F}_t$ is in $\mathcal{F}_t$ again.
You may also use a theorem from the lecture notes. Here it is important to observe the continuity of B and the closedness of the set $\{A, -A\}$.

c. Since τ is a stopping time and $(X_t)_{t\geq 0}$ a martingale, it follows from the stopping time theorem that $(X_{t\wedge\tau})_{t\geq 0}$ is a martingale again. Therefore, by the properties of the conditional expectation:

$$\mathbb{E}(X_{t\wedge\tau}) = \mathbb{E}(\mathbb{E}(X_{t\wedge\tau}|mathcal F_0)) = \mathbb{E}(X_0) = 0.$$

d. First note that $\lim_{t\to\infty} X_{t\wedge\tau}) = X_\tau$ pointwise on the set $\{\tau < \infty\}$. Observe that $|B_{t\wedge\tau}| \leq A$. Therefore,

$$|X_{t\wedge\tau}| \leq \cosh(\alpha A)\exp(-\alpha^2(t\wedge\tau)/2) \leq \cosh(\alpha A).$$

Since the constant function $\cosh(\alpha A)$ is in L^1, we may apply the dominated convergence theorem to conclude

$$1 = \lim_{t\to\infty} \mathbb{E}(X_{t\wedge\tau}) = \mathbb{E}(\lim_{t\to\infty} X_{t\wedge\tau}) = \mathbb{E}(X_\tau).$$

Next we calculate $\mathbb{E}(X_\tau)$ in a different way. Observe that on the set $\{\tau < \infty\}$ we have $|B_\tau| = A$ and hence

$$1 = \mathbb{E}(X_\tau) = \mathbb{E}(\cosh(\alpha A)\exp(-\alpha^2\tau/2)) = \cosh(\alpha A)\mathbb{E}(\exp(-\alpha^2\tau/2)).$$

Taking $\alpha^2/2 = \lambda$ we find that $\mathbb{E}(\exp(-\alpha^2\tau/2)) = 1/\cosh(\sqrt{2\lambda})$.

END

3. Let M be a continuous L^2-martingale. We will show below, in steps (a) through (c), that $M^2 - [M]$ is a martingale. Let $t > s \geq 0$. Consider a partition of $[0,t]$ given by

$$0 = t_0 < t_1 < \cdots < t_k = s < \cdots < t_n = t.$$

(4) a. Show that for $t_i \geq s$,

$$\mathbb{E}\left[M_{t_{i+1}}^2 - M_{t_i}^2 - (M_{t_{i+1}} - M_{t_i})^2 \middle| \mathcal{F}_s \right] = 0.$$

(2) b. Derive that

$$\mathbb{E}\left[M_t^2 - \sum_{i=0}^{n-1} (M_{t_{i+1}} - M_{t_i})^2 \middle| \mathcal{F}_s \right] = M_s^2 - \sum_{i=0}^{k-1} (M_{t_{i+1}} - M_{t_i})^2.$$

(3) c. Using the previous identity and the existence theorem for quadratic variations of continuous L^2-martingales show that $M^2 - [M]$ is a martingale.

(6) d. Let M and N be continuous L^2-martingales with quadratic covariation $[M,N]_t = 2t - 1$. Let $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be given by $X_t = M_t + \cos(t)$. Let $U : \mathbb{R}_+ \to \mathbb{R}$ and $V : \mathbb{R}_+ \to \mathbb{R}$ be given by $U_t = t$ and $V_t = t^2$. Use the properties of quadratic (co)variations to calculate the $[U \cdot X, V \cdot N]_t$ for $t \geq 0$.

See next page.

[Soln]

a. The tower property for $s \leq t_i$ gives that $\mathbb{E}(\cdot|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(\cdot|\mathcal{F}_{t_i})|\mathcal{F}_s)$. Thus by the definition of a martingale we can write

$$\begin{aligned}\mathbb{E}(M_{t_{i+1}}^2 - M_{t_i}^2 - (M_{t_{i+1}} - M_{t_i})^2|\mathcal{F}_s) &= \mathbb{E}(2M_{t_i}(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_s)\\ &= \mathbb{E}(\mathbb{E}(2M_{t_i}(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_{t_i})|\mathcal{F}_s)\\ &= \mathbb{E}(2M_{t_i}\mathbb{E}((M_{t_{i+1}} - M_{t_i})|\mathcal{F}_{t_i})|\mathcal{F}_s)\\ &= \mathbb{E}(2M_{t_i}0|\mathcal{F}_s) = 0.\end{aligned}$$

b. Note that $M_t^2 = M_s^2 + \sum_{i=k}^{n-1}(M_{t_{i+1}}^2 - M_{t_i}^2)$. Therefore, we find

$$\begin{aligned}\mathbb{E}\left[M_t^2 - \sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2 \,\middle|\, \mathcal{F}_s\right] &- \left(M_s^2 - \sum_{i=0}^{k-1}(M_{t_{i+1}} - M_{t_i})^2\right)\\ &= \mathbb{E}\left[M_t^2 - M_s^2 - \sum_{i=k}^{n-1}(M_{t_{i+1}} - M_{t_i})^2 \,\middle|\, \mathcal{F}_s\right]\\ &= \sum_{i=k}^{n-1}\mathbb{E}\left[M_{t_{i+1}}^2 - M_{t_i}^2 - (M_{t_{i+1}} - M_{t_i})^2 \,\middle|\, \mathcal{F}_s\right] = 0\end{aligned}$$

where in the last step we used part a.

c. The existence theorem for quadratic variations yields that for a sequence of partitions $(\pi^m)_{m\geq 1}$ of $[0,t]$ with $\text{mesh}(\pi^m) \to 0$ we have with $\pi^m = \{t_0^m, \ldots, t_{n_m}^m\}$ with $t_0^m = 0$ and $t_{n_m}^m = t$ that

$$\sum_{i=0}^{n_m-1}(M_{t_{i+1}^m} - M_{t_i^m})^2 \to [M]_t \ \text{ in } L^1.$$

Without loss of generality we can assume $s \in \pi^m$ for all m. By the contractivity of the conditional expectation (which means $\|\mathbb{E}(X|\mathcal{F}_s)\|_{L^1} \leq \|X\|_{L^1}$) we also find that

$$\mathbb{E}\left[M_t^2 - \sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2 \,\middle|\, \mathcal{F}_s\right] \to M_t^2 - [M]_t \text{ in } L^1.$$

Similarly, letting k_m be such that $t_{k_m} = s$, we find

$$M_s^2 - \sum_{i=0}^{k_m-1}(M_{t_{i+1}^m} - M_{t_i^m})^2 \to M_s^2 - [M]_s \ \text{ in } L^1.$$

d. First note that $[U \cdot X, V\dot{N}] = UV \cdot [X,N]$. Also $[\cos(\cdot), N] = 0$ because $\cos(\cdot)$ is of bounded variation. Thus $[X,N] = [M,N] + [\cos(\cdot), N] = [M,N]$ and we find

$$[U \cdot X, V\dot{N}]_t = (UV \cdot [M,N])_t = \int_0^t s^3(2s-1)\,ds = \int_0^t 2s^4 - s^3\,ds = \frac{2t^5}{5} - \frac{t^4}{4}.$$

END

4. Let (B_t) be a standard Brownian motion defined on a (filtered) probability space $(\Omega, \mathcal{F}\ (\mathcal{F}_t), P)$. Suppose X_t is a process satisfying the SDE

$$dX_t = X_t dB_t, \quad X_0 = 1. \qquad (*)$$

Define the process $(Z_t)_{t\in[0,1]}$ by

$$Z_t = X_t e^{-\int_0^t B_s^2 ds}.$$

(6) a. Apply Itô's formula to show that

$$dZ_t = Z_t(dB_t - B_t^2 dt), \quad Z_0 = 1. \qquad (**)$$

(5) b. Find the solution X_t satisfying the SDE $(*)$.

Hint: You may propose a solution and appeal to the uniqueness theorem.

(3) c. Use the solution in (b) to show that $\mathbb{E}(Z_t^2) \leq e^t$.

Hint: You may use the identity $\mathbb{E}(\exp(\xi)) = \exp(\sigma^2/2)$ for $\xi \sim N(0, \sigma^2)$.

(6) d. In the "integrated version" of the SDE $(**)$ two relevant random variables are $\int_0^1 Z_t dB_t$ and $\int_0^1 Z_t B_t^2 dt$. Use (c) to show that both of the following hold.

$$(i)\ \mathbb{E}\left[\left(\int_0^1 Z_t dB_t\right)^2\right] < \infty \quad \text{and} \quad (ii)\ \mathbb{E}\left[\int_0^1 |Z_t B_t^2| dt\right] < \infty.$$

Hint: If needed, you may assume the expressions for the moments of Gaussian distribution, without proof. For example, $\mathbb{E}(\xi^4) = 3\sigma^4$ for $\xi \sim N(0, \sigma^2)$.

[Soln]

a. Note that $(*)$ implies that X_t is a continuous process satisfying

$$(i)\quad \int_0^t X_s^2\,ds < \infty \text{ a.s.} \quad\text{ and }\quad (ii)\quad X_t = 1 + \int_0^t X_s\,dB_s.$$

Property (i) implies that $\int_0^t X_s\,dB_s$ is a (continuous) local martingale and so is X_t from (ii). Also, $Y_t := \int_0^t B_s^2\,ds$, being an increaisng function of t, is a FV process and hence is a semimartingale. Note that $Z_t = f(X_t, Y_t)$ where $f(x,y) = xe^{-y}$. Clearly, $Z_0 = X_0 = 1$ and $f \in C^2(\mathbb{R}^2)$, with

$$f_x(x,y) = e^{-y} = -f_{xy}(x,y); \quad f_{xx} = 0; \quad f_y(x,y) = -xe^{-y} = -f(x,y) \quad\text{and}\quad f_{yy} = f.$$

Applying (vector-valued) Itô's formula to $Z_t = f(X_t, Y_t)$ we have

$$dZ_t = d[f(X_t,Y_t)] = f_x dX_t + f_y dY_t + \frac{1}{2} f_{xx} d[X]_t + f_{xy} d[X,Y]_t + \frac{1}{2} f_{yy} d[Y]_t.$$

Since Y_t is a FV process, the quadratic (co)variation processes $[X,Y]_t = [Y]_t = 0$. Noting further that $dY_t = B_t^2\,dt$, from $(*)$ we have,

$$\begin{aligned} dZ_t &= f_x(X_t,Y_t)dX_t + f_y(X_t,Y_t)dY_t \;=\; e^{-Y_t}dX_t - f(X_t,Y_t)dY_t \\ &= e^{-Y_t}X_t dB_t - Z_t B_t^2 dt \;=\; Z_t(dB_t - B_t^2 dt). \end{aligned}$$

b. Integrating factor method or an educated guess (together with an application of Itô formula) shows that

$$X_t = e^{B_t - \frac{1}{2}t}$$

is **a** solution to the SDE $(*)$. The uniqueness theorem ensures that this is the (unique) solution. Note that the uniqueness theorem is applicable because the coefficient of the SDE is $b(x) = x$ and it satisfies both the Lipschitz and Growth conditions:

$$|b(x) - b(y)| \le K|x-y| \quad\text{and}\quad |b(x)|^2 \le K(1+|x|^2) \quad\text{with } K = 1.$$

c. From nonegativity of X_t and $Y_t := \int_0^t B_s^2\,ds$ it follows that, $0 \le Z_t = X_t e^{-Y_t} \le X_t$ and as a result,

$$\mathbb{E}(Z_t^2) \le \mathbb{E}(X_t^2) = \mathbb{E}[e^{2B_t - t}] = e^{-t} e^{\frac{1}{2}4t} = e^t.$$

d. Note that $Z_t \in \mathcal{L}^2(B)$, because, from part (c),

$$\int_0^1 \mathbb{E}(Z_t^2)dt \le \int_0^1 e^t dt \le (e-1) < \infty.$$

We can then apply Itô isometry and this leads to (i):

$$\mathbb{E}\left[\left(\int_0^1 Z_t dB_t\right)^2\right] = \int_0^1 \mathbb{E}(Z_t^2)dt \le (e-1) < \infty.$$

For (ii), note that using Fubini and (two times) Cauchy-Schwartz we have

$$\begin{aligned} \mathbb{E}\left[\int_0^1 |Z_t B_t^2|\,dt\right] &= \int_0^1 E[|Z_t B_t^2|]dt \;\le\; \int_0^1 (E[Z_t^2])^{\frac{1}{2}} (E[B_t^4])^{\frac{1}{2}} dt \\ &\le \left(\int_0^1 E[Z_t^2]dt\right)^{\frac{1}{2}} \left(\int_0^1 E[B_t^4]dt\right)^{\frac{1}{2}} \;\le\; (e-1)^{\frac{1}{2}} \left(\int_0^1 3t^2 dt\right)^{\frac{1}{2}} \\ &= \sqrt{e-1} \;<\; \infty. \end{aligned}$$

END

5. Let $0 < T < \infty$. Suppose H is a predictable process satisfying

$$\int_0^T H_t^2\,dt < \infty, \quad \text{a.s.} \qquad (\dagger)$$

and let $Z_t \equiv Z_t(H)$ be given by

$$Z_t(H) = \exp\left\{\int_0^t H_s\,dB_s - \frac{1}{2}\int_0^t H_s^2\,ds\right\}.$$

(5) a. Use Itô's formula to Z to show that, under $(\dagger)$, Z is a continuous local martingale.

In the Girsanov Theorem an important assumption is that $(Z_t)_{t\in[0,T]}$ is a martingale. Since Z is nonnegative using Question (1c) one sees that Z is a supermartingale. It can be shown that $(Z_t)_{t\in[0,T]}$ is a martingale if and only if $\mathbb{E}[Z_T] = 1$. In this exercise we will check the latter in several steps under the assumption that there is a constant C such that

$$\int_0^T H_t^2\,dt \;\le\; C, \quad \text{a.s.} \qquad (\dagger\dagger)$$

Let $\sigma_n = \inf\{0 \le t \le T : \; Z_t \ge n\}$, where we set $\sigma_n = T$ if the set is empty. In this way $(\sigma_n)_{n\ge1}$ is a localizing sequence for Z. Denote by $Z_t^{(n)}$ the stopped process $Z_{t\wedge\sigma_n}$. It then follows immediately that

$$\mathbb{E}\big(Z_t^{(n)}\big) = 1, \quad n \ge 1,\; 0 \le t \le T.$$

(3) b. Show from the definitions that $Z_t^{(n)} = Z_t(H^{(n)})$ a.s., where $H^{(n)}$ is given by

$$H_s^{(n)}(\omega) = H_s(\omega)\mathbf{1}_{[0,\sigma_n(\omega)]}(s), \qquad \text{for } 0 \le s \le T.$$

(5) c. Show that if $(\dagger\dagger)$ holds then $\mathbb{E}\Big(\big(Z_T^{(n)}\big)^2\Big) \le e^C$, for all $n \ge 1$.

Hint: Argue that $\Big(Z_t^{(n)}\Big)^2 \le M_t^{(n)}\,e^C$ for some suitable martingale $M_t^{(n)}$ with $M_0^{(n)} = 1$.

(4) d. Show that, under $(\dagger\dagger)$, $\mathbb{E}[Z_T] = 1$.

Hint: Argue that $\sigma_n \to T$ a.s. and $Z_T^{(n)} \to Z_T$ a.s. Use Question 1 to conclude that $\{Z_T^{(n)},\; n \ge 1\}$ is uniformly integrable. Recall further that for $X, X_n \in L^1$ $(n \ge 1)$, $X_n \to X$ in L^1 if and only if (i) $X_n \to X$ in probability and (ii) $\{X_n,\; n \ge 1\}$ is uniformly integrable.

[Soln]

a. Note that under (†), $M_t := \int_0^t H_s\, dB_s$ is a local martingale and $N_t := \int_0^t H_s^2\, ds$ is a FV process on $[0,T]$. In particular, both of them are semimartingales. Since $Z_t \equiv Z_t(H) = f(M_t, N_t)$ where $f(x,y) = e^{x-\frac{1}{2}y}$ and both M and N are continous, it follows that Z_t is also continuous. Noting that $f \in C^2(\mathbb{R}^2)$, we can apply (vector-valued) Itô formula to $Z_t = f(M_t, N_t)$ to obtain

$$\begin{aligned} dZ_t &= f_x dM_t + f_y dN_t + \frac{1}{2} f_{xx} d[M]_t + f_{xy} d[M,N]_t + \frac{1}{2} f_{yy} d[N]_t \\ &= f_x dM_t + f_y dN_t + \frac{1}{2} f_{xx} d[M]_t, \quad \text{since } [N]_t = [M,N]_t = 0. \end{aligned}$$

Noting that $[M]_t = \int_0^t H_s^2\, ds = N_t$ and $f_x = f_{xx} = f = -2f_y$, we get

$$Z_t \;=\; Z_0 + \int_0^t f(M_s, N_s)\, dM_s \;=\; 1 + \int_0^t Z_s\, dM_s \;=\; 1 + \int_0^t Z_s\, H_s\, dB_s.$$

Now, almost sure continuity of Z implies that

$$\int_0^t (Z_s\, H_s)^2\, ds \;\le\; \left(\sup_{s\in[0,T]} Z_s \right) \int_0^t H_s^2\, ds < \infty \quad \text{(a.s.)}$$

This in turn implies that $\int_0^t Z_s\, H_s\, dB_s$ and hence $Z_t = 1 + \int_0^t Z_s\, H_s\, dB_s$ is a (continuous) local martingale.

b. Let $n \ge 1$. Then

$$\begin{aligned} Z_t^{(n)} &= Z_{t\wedge\sigma_n} = e^{\int_0^{t\wedge\sigma_n} H_s\, dB_s - \frac{1}{2}\int_0^{t\wedge\sigma_n} H_s^2\, ds} = e^{\int_0^t H_s \mathbf{1}_{[0,\,\sigma_n]}(s)\, dB_s - \frac{1}{2}\int_0^t H_s^2 \mathbf{1}_{[0,\,\sigma_n]}(s)\, ds} \\ &= e^{\int_0^t H_s^{(n)}\, dB_s - \frac{1}{2}\int_0^t (H_s^{(n)})^2\, ds} = Z_t(H^{(n)}), \quad \text{from the definition of } Z(H). \end{aligned}$$

c. First note that, since $H^{(n)}$ satisfies (†), it follows from part (a) that $Z_t^{(n)} = Z_t(H^{(n)})$ is a continuous local martingale. But it is also nonnegative and hence by Question 1(c), it is a supermartingale. The fact that $E\left(Z_t^{(n)}\right) = 1$ then implies that $Z_t^{(n)}$ is a martingale. The same argument will also lead to the conclusion that $Z_t(2H^{(n)})$ is a martingale.

Next note that from (††) it follows that

$$\begin{aligned} \left(Z_t^{(n)}\right)^2 &= e^{2\int_0^t H_s^{(n)}\, dB_s - \int_0^t (H_s^{(n)})^2\, ds} = e^{\int_0^t 2H_s^{(n)}\, dB_s - \frac{1}{2}\int_0^t (2H_s^{(n)})^2\, ds + \int_0^t (H_s^{(n)})^2\, ds} \\ &= Z_t(2H^{(n)})\, e^{\int_0^t (H_s^{(n)})^2\, ds} \;\le\; Z_t(2H^{(n)})\, e^C, \end{aligned}$$

where, as mentioned earlier, $Z_t(2H^{(n)})$ is a martingale. Hence

$$E\left[\left(Z_T^{(n)}\right)^2\right] \;\le\; E\left[Z_t(2H^{(n)})\right] e^C \;=\; E\left[Z_0(2H^{(n)})\right] e^C \;=\; e^C.$$

d. Almost sure continuity of Z implies that $\exists\, \Omega_0$ with $P(\Omega_0) = 1$ such that $Z_t(\omega)$ is continuous in t for each $\omega \in \Omega_0$. Then for $\omega \in \Omega_0$, $Z^*(\omega) := \sup_{t\in[0,T]} Z_t(\omega) < \infty$. Note that for all $n \ge Z^*(\omega)$, $\sigma_n(\omega) = T$. Hence $\lim_{n\to\infty} \sigma_n(\omega) = T$ for all $\omega \in \Omega_0$. In other words, $\sigma_n \to T$ (a.s.), as $n \to \infty$. Consequently, $\lim_{n\to\infty} Z_T^{(n)} = \lim_{n\to\infty} Z_{T\wedge\sigma_n} = Z_T$ (a.s.) In particular, $Z_T^{(n)}$ converges to Z_T in probability. The L^2-boundedness of $\{Z_T^{(n)},\ n \ge 1\}$, obtained in part (c), together with Question 1(b), imply uniform integrability of the sequence. Uniform integrability and convergence in prob imply that $Z_T^{(n)}$ converges to Z_T in L^1. In particular, $E[Z_T] = \lim_{n\to\infty} E\left[Z_T^{(n)}\right]$. But $E\left[Z_T^{(n)}\right] = 1$ for $n \ge 1$. Hence, $E[Z_T] = 1$.

END

Some comments on grading of Question 4 and 5

- Before applying any theorem or formula in/to a situation, you should make sure that your situation satisfies all conditions stated in the statement of the theorem/formula. For example,
  - Ito isometry: integrand must be in $\mathcal{L}^2(B)$
  - Ito formula: function $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$.

    Note that function f should be deterministic, as the notations suggest, and should not involve ω (randomness). $f(t,x) = x\exp(\int_0^t H_s^2\,ds)$ where H_s is a stochastic process is not an appropriate function to apply Ito formula to.

    Also, it is better to write out the full/complete formula and then fill-in the values (e.g., $f_{xx} = 0$ or $[X,Y]_t = 0$).
  - Uniqueness theorem of solution: coefficients must satisfy growth and Lipschitz conditions.

- In (5a) one needs, as intermediate step, $\int_0^t Z_s H_s\,dB_s$ to be a local martingale. This needs proof. One way is to check that $\int_0^t Z_s^2 H_s^2\,ds < \infty$ a.s. Then the integral makes sense and is a local martingale.

END