1. **[Soln]**

   a. Let $(\tau_n)_{n\geq 0}$ be a localizing sequence for the local martingale M, i.e. for all $s,t \in \mathbb{R}_+, s<t$ we have
   $$\mathbb{E}(M_{\tau_n \wedge t}|\mathcal{F}_s) = M_{\tau_n \wedge s}. \tag{1}$$
   Since $\sup_{s\geq 0}|M_s| \in L^1$, we have that for all n,t, $|M_{\tau_n\wedge t}| \leq \sup_{s\geq 0}|M_s|$ is bounded by an integrable random variable. By dominated convergence we can pull the limit in (1) out and get the result. [Now taking the limit as $n\to\infty$ in (1) the result follows from the dominated convergence theorem.]

   b. For the definition see the book.
   Now call $A_k = \{|\mathbb{E}(X|\mathcal{G})| > k\}$. We have to show that given $\epsilon > 0$, $\exists\, K$ such that
   $$\sup_{\mathcal{G}\subset\mathcal{F}} \mathbb{E}(1_{A_k}|\mathbb{E}(X|\mathcal{G})|) < \epsilon \qquad \forall\, k \geq K.$$
   Since $X \in L^1$, we have that for a given $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathbb{P}(A) \leq \delta$ we have $\mathbb{E}(|X|1_A) \leq \epsilon$. Choose K large enough such that $K \geq \mathbb{E}(|X|)/\delta$. By Chebyshev's and Jensen inequality for $k \geq K$
   $$\mathbb{P}(A_k) \leq \frac{1}{k}\mathbb{E}(|\mathbb{E}(X|\mathcal{G})|) \leq \frac{1}{k}\mathbb{E}(|X|) \leq \delta.$$
   Since $A_k \in \mathcal{G}$ the result follows from
   $$\mathbb{E}(|\mathbb{E}(X|\mathcal{G})|1_{A_k}) \leq \mathbb{E}(\mathbb{E}(|X||\mathcal{G})1_{A_k}) = \mathbb{E}(|X|1_{A_k}) \leq \epsilon.$$

   c. i) From Itô-isometry
   $$\begin{aligned}(H\bullet(K\bullet M)_t) &= \int_0^t H_s d\left(\int_0^s K_u dM_u\right)\\ &= \int_0^t H_s(K_s dM_s - K_0 dM_0) = ((HK)\bullet M)_t\end{aligned}$$

   ii) Pick $N = \int_0^\cdot H_s dM_s$, then
   $$\left\langle \int_0^\cdot H_s dM_s, \int_0^\cdot H_s dM_s \right\rangle_t = \int_0^t H_s d\left\langle M, \int_0^\cdot H_u dM_u\right\rangle_s = \int_0^t H_s^2 d\langle M\rangle_s.$$

2. **[Soln]**

First note that, for all $t \geq 0$,

$$\int_0^t (\sin(H_s))^2 \, ds \leq t \quad \text{and} \quad \int_0^t (\cos(H_s))^2 \, ds \leq t.$$

Hence the stochastic integrals w.r.t. BM with $\sin(H_s)$ and $\cos(H_s)$ as integrands, i.e.,

$$\int_0^t \cos(H_s)\, dX_s, \quad \int_0^t \sin(H_s)\, dY_s, \quad \int_0^t \sin(H_s)\, dX_s \quad \text{and} \quad \int_0^t \cos(H_s)\, dY_s$$

are all local martingales. In fact, they are all L^2-martingales. Also, clearly they are continuous. Hence

$$B_t = \int_0^t \cos(H_s)\, dX_s + \int_0^t \sin(H_s)\, dY_s$$
$$\hat{B}_t = \int_0^t \sin(H_s)\, dX_s - \int_0^t \cos(H_s)\, dY_s.$$

are also continuous local martingales.

From Levy characterization theorem, it will follow that continuous local martingales B and $\hat{B}$ are two independent Brownian motions, if we show that the quadratic variation processes are given by

$$\langle B\rangle_t \;=\; \langle \hat{B}\rangle_t \;=\; t, \qquad \langle B\,,\,\hat{B}\rangle_t \;=\; 0.$$

But, since X and Y are independent Brownian motions, we have

$$\begin{aligned}
\langle B\rangle_t \;&=\; \left\langle \int_0^{(\cdot)} \cos(H_s)\, dX_s \right\rangle_t + \left\langle \int_0^{(\cdot)} \sin(H_s)\, dY_s \right\rangle_t + \left\langle \int_0^{(\cdot)} \cos(H_s)\, dX_s\,,\, \int_0^{(\cdot)} \sin(H_s)\, dY_s \right\rangle_t \\
&=\; \int_0^t (\cos(H_s))^2\, d\langle X\rangle_s + \int_0^t (\sin(H_s))^2\, d\langle Y\rangle_s + \int_0^t \cos(H_s)\, \sin(H_s)\, d\langle X\,,\,Y\rangle_s \\
&=\; \int_0^t (\cos(H_s))^2\, ds + \int_0^t (\sin(H_s))^2\, ds + 0 \;=\; \int_0^t ds \\
&=\; t.
\end{aligned}$$

Similarly, we can show that $\langle \hat{B}\rangle_t \;=\; t$. Finally,

$$\begin{aligned}
\langle B\,,\,\hat{B}\rangle_t \;&=\; \int_0^t \cos(H_s)\, \sin(H_s)\, d\langle X\rangle_s - \int_0^t (\cos(H_s))^2\, d\langle X\,,\,Y\rangle_s \\
&\quad + \int_0^t (\sin(H_s))^2\, d\langle Y\,,\,X\rangle_s - \int_0^t \sin(H_s)\, \cos(H_s)\, d\langle Y\rangle_s \\
&=\; \int_0^t \cos(H_s)\, \sin(H_s)\, ds - 0 + 0 - \int_0^t \sin(H_s)\, \cos(H_s)\, ds \\
&=\; 0.
\end{aligned}$$

Hence the proof is complete.

3. **[Soln]**

a.) X is continuous and starts at a point $x_0 \in (a, b)$ hence the exit time of this interval can be determined by $T_a \wedge T_b$.

b.) We can use Itô's formula to show that $f(X_t)$ and $f(X_{t\wedge T})$ are semimartingales. Note since

$$\langle X \rangle_s = \left\langle x_0 + \int_0^{\cdot} \sigma(X_u) dB_u \right\rangle_s = \int_0^s \sigma^2(X_u) du$$

the decompositions are equal to

$$f(X_t) = f(x_0) + \int_0^t f'(X_s)\sigma(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)ds$$

resp. analogous for $f(X_{t\wedge T})$ which is a combination of a local martingale and stochastic integral w.r.t. the Lebesgue measure.

c.) First we show that for all $t > 0$, $\langle M \rangle_t \in L^1$ by upper bounding

$$\left\langle \int_0^{\cdot\wedge T} f'(X_s)\sigma(X_s)dB_s \right\rangle_t \leq \int_0^t \sup_{0\leq u\leq T} (f')^2(X_u)\sigma^2(X_u)du \leq Ct$$

where the last inequality follows from the fact that σ is continuous and $f \in C^2$. Since M is a martingale, via Doob's decomposition we get that $(M_t^2 - \langle M \rangle_t)_{t\geq 0}$ is also a martingale and $\mathbb{E}(M_t^2) = \mathbb{E}(\langle M \rangle_t) < \infty$, hence $M \in L^2$.

d.) Direct consequence.

e.) Trivial.

f.) Follows from

$$\mathbb{E}(h(X_{t\wedge T})) = h(x_0) + \frac{1}{2}\mathbb{E}\left(\int_0^{t\wedge T} h''(X_s)\sigma^2(X_s)ds\right) = h(x_0) + \frac{1}{2}\mathbb{E}\left(\int_0^{t\wedge T} 1ds\right).$$

g.) It is trivial to show that $t \wedge T \in L^1$ and since $t \wedge T \to T$ almost surely, we have by monotone convergence that $T \in L^2$

h.) Trivial.

i.) From (d) we get

$$\begin{aligned}\mathbb{E}(w(X_{t\wedge T})) &= w(x_0) + \frac{1}{2}\mathbb{E}\left(\int_0^{t\wedge T} w''(X_s)\sigma^2(X_s)ds\right) = w(x_0) + \frac{1}{2}\mathbb{E}(t\wedge T) \\ &\overset{(f)}{=} \mathbb{E}(h(X_{t\wedge T})) - \frac{b - x_0}{b - a}\end{aligned}$$

we can take the limit $t \to \infty$ and get

$$\mathbb{E}(w(X_T)) = \mathbb{E}(h(X_{t\wedge T})) + \frac{b - x_0}{b - a} = \frac{b - x_0}{b - a}$$

We use that $w(X_T) = h(X_T) + \frac{b - X_T}{b - a}$ and solve the above equation

$$\mathbb{E}(X_T) = x_0.$$

The claim follows from $\mathbb{E}(X_T) = a\mathbb{P}(X_T = a) + b\mathbb{P}(X_T = b)$ and $\mathbb{P}(X_T = a) + \mathbb{P}(X_T = b) = 1$.

4. **[Soln]**

a. We are going to consider solutions of the form

$$X_t = a(t) \left\{ x_0 + \int_0^t b(s)\, dB_s \right\}.$$

Let

(i) $Y_t = x_0 + \int_0^t b(s)\, dB_s$ be a local martingale and

(ii) $f(t,x) := a(t)\, x \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$.

Note that $X_t = f(t, Y_t)$ and

$$f_t(t,x) = a'(t)\, x = \frac{a'(t)}{a(t)}\, f(t,x), \quad f_x(t,x) = a(t), \quad f_{xx}(t,x) = 0.$$

Using Itô formula we have

$$\begin{aligned} dX_t &= d[f(t,Y_t)] = f_t(t,Y_t)\, dt + f_x(t,Y_t)\, dY_t + \frac{1}{2} f_{xx}(t,Y_t)\, d[Y]_t \\ &= \frac{a'(t)}{a(t)}\, f(t,Y_t)\, dt + a(t)\, dY_t + 0 = \frac{a'(t)}{a(t)}\, X_t\, dt + a(t)\, b(t)\, dB_t. \end{aligned}$$

So, for X to be a solution to the SDE

$$dX_t = -\alpha\, X_t\, dt + \sigma\, dB_t$$

it suffices to take

$$\frac{a'(t)}{a(t)} = -\alpha, \text{ i.e., } a(t) = c\, e^{-\alpha\, t}, \text{ for some constant } c, \quad \text{and} \quad a(t)\, b(t) = \sigma.$$

To satisfy the initial condition $X_0 = x_0$ we need

$$x_0 = f(0, Y_0) = a(0)\, x_0 \quad \Leftarrow \quad a(0) = 1 \quad \Leftrightarrow \quad c = 1.$$

Hence we consider $a(t) = e^{-\alpha\, t}$ and $b(t) = \sigma / a(t) = \sigma\, e^{\alpha\, t}$.
Note that with this choice of $a(\cdot)$ and $b(\cdot)$, both (i) and (ii) are satisfied. In fact, Y is a L^2-martingale, because it is a stochastic integral w.r.t. the Brownian motion where the integrand satisfies

$$\int_0^t b(s)^2\, ds = \int_0^t \sigma^2\, e^{\alpha\, s}\, ds = \frac{\sigma^2}{\alpha}\, \left(e^{\alpha\, t} - 1 \right) < \infty, \quad \forall\, t.$$

Hence a solution to the SDE is given by

$$X_t = a(t) \left\{ x_0 + \int_0^t b(s)\, dB_s \right\} = x_0\, e^{-\alpha\, t} + \sigma \int_0^t e^{-\alpha\, (t-s)}\, dB_s.$$

From the unqueness of solution it follows that this is the only/unique solution.

b. We now consider the process

$$Y_t = \exp\left\{ X_t - \frac{\eta}{\alpha} \right\} = g(X_t), \text{ say,} \quad \text{where } g(x) = e^{x - \eta/\alpha}.$$

Clearly $g \in C^2(\mathbb{R})$, with $g' = g'' = g$. Using Itô formula we have

$$dY_t = d[g(X_t)] = g'(X_t)\, dX_t + \frac{1}{2}\, g''(X_t)\, d[X]_t = Y_t\, [-\alpha\, X_t\, dt + \sigma\, dB_t] + \frac{1}{2}\, Y_t\, d[X]_t.$$

From the given SDE it follows that X_t satisfies

$$X_t = x_0 - \alpha \int_0^t X_s\, ds + \sigma B_t.$$

Since the second term in the above expression is of finite variation (a.s.), the quadratic variation of X is given by $[X]_t = \sigma^2\,[B]_t = \sigma^2\,t$. We then get the SDE satisfied by Y_t:

$$\begin{aligned}
dY_t &= -\alpha\,Y_t\,X_t\,dt + \sigma\,Y_t\,dB_t + \frac{1}{2}\,Y_t\,\sigma^2\,dt \\
&= Y_t\left[-\alpha\,X_t + \frac{1}{2}\,\sigma^2\right]dt + \sigma\,Y_t\,dB_t \\
&= Y_t\left[-\alpha\,\left(\ln Y_t + \frac{\eta}{\alpha}\right) + \frac{1}{2}\,\sigma^2\right]dt + \sigma\,Y_t\,dB_t \\
&= Y_t\left(\frac{1}{2}\,\sigma^2 - \eta - \alpha\,\ln Y_t\right)dt + \sigma\,Y_t\,dB_t, \quad t > 0
\end{aligned}$$

with $Y_0 = e^{x_0 - \eta/\alpha}$.

c. Now consider the SDE satisfied by a geometric mean reverting process:

$$dr_t = r_t\,(\theta - \alpha\,\ln r_t)\,dt + \sigma\,r_t\,dB_t, \quad t > 0$$

with $r_0 = 1$.
Comparing this with the SDE in (b), satisfied by Y_t, we realize that it suffices to take

$$\frac{1}{2}\,\sigma^2 - \eta = \theta, \quad \text{i.e.,} \quad \eta = \frac{1}{2}\,\sigma^2 - \theta, \qquad \text{and} \qquad x_0 = \frac{\eta}{\alpha} = \frac{\sigma^2/2 - \theta}{\alpha}.$$

Hence from (a) and (b) it follows that the required solution is given by

$$r_t = \exp\left\{\tilde{r}_t - \frac{\sigma^2/2 - \theta}{\alpha}\right\}$$

where

$$\tilde{r}_t = \frac{\sigma^2/2 - \theta}{\alpha}\,e^{-\alpha\,t} + \sigma\,\int_0^t e^{-\alpha\,(t-s)}\,dB_s.$$

5. **[Soln]**
Note that formally we can rewrite X_t under $\mathbb{P}$ as follows.

$$\begin{aligned} X_t &= x + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dB_s \\ &= x + \int_0^t \nu_s\, ds + \int_0^t (\mu_s - \nu_s)\, ds + \int_0^t \sigma_s\, dB_s \\ &= x + \int_0^t \nu_s\, ds + \int_0^t \sigma_s \left[dB_s + \frac{\mu_s - \nu_s}{\sigma_s}\, ds \right] \end{aligned}$$

Hence by defining

$$\tilde{B}_t = B_t + \int_0^t \frac{\mu_s - \nu_s}{\sigma_s}\, ds = B_t - \int_0^t H_s\, ds \quad \text{with } \; H_t = \frac{\nu_t - \mu_t}{\sigma_t},$$

we have

$$(\star) \qquad X_t = x + \int_0^t \nu_s\, ds + \int_0^t \sigma_s\, d\tilde{B}_s \quad \text{a.s. } \; [\mathbb{P}].$$

To obtain a new measure $\mathbb{Q}$, under which $\tilde{B}$ is a Brownian motion, we are going to apply Girsanov theorem. Now assume that $\sigma_t \neq 0 \; \forall\, t \in \mathbb{R}_+$, so that $H_t = \frac{\nu_t - \mu_t}{\sigma_t}$ is well defined. Clearly, H is an adpated process. To apply Girsanov theorem we need furthermmore

$$(\dagger) \qquad \int_0^T H_s^2\, ds < \infty \quad \text{a.s. } \; [\mathbb{P}],$$

and $Z_T \equiv Z_T(H)$ to be a $\mathbb{P}$-martingale, where

$$Z_t = \exp\left\{ - \int_0^t H_s\, dB_s - \frac{1}{2} \int_0^t H_s^2\, ds \right\}.$$

The martingale condition is satisfied if, for example, (ν_t), (μ_t) and (σ_t) are deterministic/non-random processes satisfying $\sigma_t \neq 0$ and $(\dagger)$.
In this case, $\tilde{B}$ is a BM under $\mathbb{Q}$, which is defined as $d\mathbb{Q} = Z_T\, d\mathbb{P}$. Since $Z_T > 0$, it follows that $\mathbb{Q} \equiv \mathbb{P}$, i.e., the null sets of both the measures are the same. From $(\star)$ it then follows that

$$X_t = x + \int_0^t \nu_s\, ds + \int_0^t \sigma_s\, d\tilde{B}_s \quad \text{a.s. } \; [\mathbb{Q}]$$

where $\tilde{B}$ is a $\mathbb{Q}$-BM.