Exam Stochastic Differential Equations (3TU)

Solutions

June 15, 2009

1. (a) $r + \sum_{i=1}^n X_i$ is the total number of red balls after the nth drawing. The total number of balls in the urn after n drawings is $b + r + n$. So the fraction of red balls in the urn after the nth drawing is given by $Z_n = (r + \sum_{i=1}^n X_i)/(b + r + n)$.

   (b) From (a) we have that $Z_n = f_n(X_1, \ldots, X_n)$ where $f_n(x_1, \ldots, x_n) = (r + \sum_{i=1}^n x_i)/(b + r + n)$. The expectation of Z_n is finite since $|Z_n| \leq 1$.

$$\begin{aligned} E(Z_n \mid X_1, \ldots, X_{n-1}) &= E\left(\frac{r + \sum_{i=1}^n X_i}{r + b + n} \mid X_1, \ldots, X_{n-1}\right) \\ &= \frac{r + \sum_{i=1}^{n-1} X_i}{r + b + n} + \frac{1}{r + b + n} E\left(X_n \mid X_1, \ldots, X_{n-1}\right). \end{aligned}$$

   Now

$$\frac{r + \sum_{i=1}^{n-1} X_i}{r + b + n} = \left(1 - \frac{1}{r + b + n}\right) Z_{n-1}$$

   and

$$E\left(X_n \mid X_1, \ldots, X_{n-1}\right) = Z_{n-1}$$

   so

$$E(Z_n \mid X_1, \ldots, X_{n-1}) = Z_{n-1}.$$

   (c) It follows from $|Z_n| \leq 1$ that $E(|Z_n|) \leq 1 < \infty$. So the statement follows from the Bounded Martingale Convergence Theorem.

   (d) The sequence Z_n is uniformly integrable since $|Z_n| \leq 1$, so Z_n converges in L^1 to Z_∞. In particular, $\lim E[Z_n] = E[Z_\infty]$. It follows from (a) that $\sum_{i=1}^n X_i = (r + b + n)Z_n - r$, so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n E[X_i] = \lim_{n\to\infty} \left(\frac{r + b}{n} + 1\right) E[Z_n] - \frac{r}{n} = E[Z_\infty].$$

2. (a) Since B_t is a martingale, we can apply Doob's Continuous-Time Stopping Theorem to conclude that $B_{t\wedge\tau}$ is a martingale. $|B_{t\wedge\tau}| \leq A + B$.

   (b) Since $P(\tau < \infty) = 1$, we have $B_\tau = \lim_{t\to\infty} B_{t\wedge\tau}$. It follows from (b) that $E(B_{t\wedge\tau}) = 0$ and that we can apply the DCT to conclude that

$$E[B_\tau] = E[\lim_{t\to\infty} B_{t\wedge\tau}] = \lim_{t\to\infty} E[B_{t\wedge\tau}] = 0.$$

   Since B_τ can only take the values A and $-B$, we get $AP(B_\tau = A) - B(1 - P(B_\tau = A)) = 0$. So $P(B_\tau = A) = B?(A + B)$ and $P(B_\tau = B) = 1 - P(B_\tau = A) = A/(A + B)$.

   (c) (i) M_t is $\mathcal{F}_t$ measurable since it is a continuous function of B_t. (ii) M_t is integrable, since $E[|M_t|] \leq E[B_t^2] + t = 2t < \infty$. (iii) Let $s < t$. Using independent increments and properties of conditional expectations, we get

$$\begin{aligned} E[M_t \mid \mathcal{F}_s] &= E[(B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 - t \mid \mathcal{F}_s] \\ &= E[(B_t - B_s)^2] + 2B_s E[B_t - B_s] + B_s^2 - t = t - s + B_s^2 - t = M_s. \end{aligned}$$

   It follows from (i), (ii) and (iii) that M_t is a martingale.

   (d) By Doob's Continuous-Time Stopping Theorem, $M_{t\wedge\tau}$ is a martingale and $|M_{t\wedge\tau}| \leq A^2 + B^2 + \tau$. So $E[M_{t\wedge\tau}] = E[M_0] = 0$. Since the upperbound $A^2 + B^2 + \tau$ is an integrable random variable it follows by the DCT that

$$E[M_\tau] = E[\lim_{t\to\infty} M_{t\wedge\tau}] = \lim_{t\to\infty} E[M_{t\wedge\tau}] = 0,$$

   hence $E[\tau] = E[B_\tau^2] = A^2 P(B_\tau = A) + B^2 P(B_\tau = B) = \frac{A^2B}{A+B} + \frac{AB^2}{A+B} = AB$.

3. (a) $\mathrm{d}X_t = t\mathrm{d}B_t$ and $\mathrm{d}Y_t = B_t\mathrm{d}t$, so by Ito's product rule

$$\mathrm{d}X_tY_t = X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t = B_tX_t\mathrm{d}t + tY_t\mathrm{d}B_t.$$

   (b) Integrating the stochastic differential derived in (a) over $[0, t]$, we get

$$X_tY_t = \int_0^t B_sX_s\mathrm{d}s + \int_0^t sY_s\mathrm{d}B_s.$$

   Note that

$$E[Y_t^2] = E\left[\int_0^t \int_0^t B_uB_v\,\mathrm{d}u\mathrm{d}v\right] = \int_0^t \int_0^t u \wedge v\,\mathrm{d}u\mathrm{d}u = \frac{1}{3}t^3.$$

   So

$$E\int_0^T (tY_t)^2\,\mathrm{d}t = \int_0^T \frac{1}{3}t^5\,\mathrm{d}t = T^6/18 < \infty,$$

and the integrand tY_t is in $\mathcal{H}^2$. It follows that $E\left[\int_0^t sY_s \mathrm{d}B_s\right] = 0$ and

$$\mathrm{Cov}(X_t, Y_t) = E[X_tY_t] = E\left[\int_0^t B_sX_s \mathrm{d}s\right].$$

So we have to calculate $E[B_sX_s]$.

$$\mathrm{d}B_tX_t = X_t\mathrm{d}B_t + B_t\mathrm{d}X_t + \mathrm{d}B_t\mathrm{d}X_t = (X_t + tB_t)\mathrm{d}B_t + t\mathrm{d}t$$

Integrating and taking expectations, we get $E[B_sX_s] = \int_0^s t\mathrm{d}t = \frac{1}{2}s^2$. It follows that

$$\mathrm{Cov}(X_t, Y_t) = \int_0^t E\left[B_sX_s\right] \mathrm{d}s = \int_0^t \frac{1}{2}s^2 \mathrm{d}s = \frac{1}{6}t^3.$$

4. (a) (Z_t) satisfies the stochastic differential equation (SDE)

$$\mathrm{d}Z_t = -aZ_t\, \mathrm{d}t + \sigma\, \mathrm{d}B_t.$$

To solve the SDE, apply Itô's formula to $e^{at}Z_t$

$$\mathrm{d}e^{at}Z_t = e^{at}\, \mathrm{d}Z_t + ae^{at}Z_t\, \mathrm{d} = \sigma e^{at}\, \mathrm{d}B_t.$$

Integrating over $[0, t]$ and substituting the initial condition $Z_0 = y_0 - \frac{\theta}{a}$ we get

$$e^{at}Z_t - \left(y_0 - \frac{\theta}{a}\right) = \sigma \int_0^t e^{as}\, \mathrm{d}B_s.$$

It follows that

$$Y_t = \frac{\theta}{a} + \left(y_0 - \frac{\theta}{a}\right)e^{-at} + \sigma \int_0^t e^{-a(t-s)}\, \mathrm{d}B_s.$$

(b) By an application of Itô's formula

$$\mathrm{d}e^{Y_t} = e^{Y_t}\, \mathrm{d}Y_t + \frac{1}{2}e^{Y_t}(\mathrm{d}Y_t)^2 = e^{Y_t}\left[(\theta - aY_t) + \frac{1}{2}\sigma^2\right]\, \mathrm{d}t + \sigma e^{Y_t}\, \mathrm{d}B_t.$$

It follows that (X_t) is a solution of the SDE

$$\mathrm{d}Z_t = Z_t\left[\frac{1}{2}\sigma^2 + (\theta - ay_0)e^{-at} - a\sigma \int_0^t e^{-a(t-s)}\, \mathrm{d}B_s\right]\, \mathrm{d}t + \sigma Z_t\, \mathrm{d}B_t.$$

(c) By an application of Itô's formula

$$\mathrm{d}\log r_t = \frac{1}{r_t}\,\mathrm{d}r_t - \frac{1}{2}\frac{1}{r_t^2}(\mathrm{d}r_t)^2 = (\eta - \frac{1}{2}\sigma^2 - a\log r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t.$$

It follows from (a) that

$$\log r_t - \log r_0 = \frac{\eta - \frac{1}{2}\sigma^2}{a} + \left(y_0 - \frac{\eta - \frac{1}{2}\sigma^2}{a}\right)e^{-at} + \sigma\int_0^t e^{-a(t-s)}\,\mathrm{d}B_s,$$

and

$$r_t = r_0\exp\left\{\frac{\eta - \frac{1}{2}\sigma^2}{a} + \left(y_0 - \frac{\eta - \frac{1}{2}\sigma^2}{a}\right)e^{-at} + \sigma\int_0^t e^{-a(t-s)}\,\mathrm{d}B_s\right\}.$$

5. Note first that $Y_t = y_0e^{rt}$, so $\tilde{X}_t = y_0^{-1}e^{-rt}X_t$. Since

$$\mathrm{d}e^{-rt}X_t = -re^{-rt}X_t\,\mathrm{d}t + e^{-rt}\,\mathrm{d}X_t = e^{-rt}X_t[(\mu - r)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t],$$

we may conclude

$$\mathrm{d}\tilde{X}_t = \sigma\tilde{X}_t\left[\frac{\mu - r}{\sigma}\,\mathrm{d}t + \mathrm{d}B_t\right] = \sigma\tilde{X}_t\,\mathrm{d}\tilde{B}_t,$$

where $\tilde{B}_t = B_t + \frac{\mu - r}{\sigma}t$. Let Q be the measure defined by

$$Q(A) = E_P\left[\mathbf{1}_A\exp\left(-\frac{\mu - r}{\sigma}B_T - \left(\frac{\mu - r}{\sigma}\right)^2 T/2\right)\right].$$

It follows from Girsanov's Theorem that $(\tilde{B}_t)$ is standard Brownian motion, which implies that $(\tilde{X}_t)$ is a Q-martingale.