EXAM STOCHASTIC DIFFERENTIAL EQUATIONS (3TU) – SOLUTIONS
May 30, 2011

Grading: $2 + (1{+}1\frac{1}{2}) + 1\frac{1}{2} + 2 + 2$

1. Since $\sigma(\gamma_1)$ is the collection of all sets of the form $\{\gamma_1 \in A\}$ with A a Borel set in $\mathbb{R}$, we must show that

$$\int_{\{\gamma_1 \in A\}} 1_{\{\gamma_1 \geq \gamma_2\}}\, dP = \int_{\{\gamma_1 \in A\}} \Psi(\gamma_1)\, d\mathbb{P}$$

By independence, the joint density of (γ_1, γ_2) is given by

$$f_{(\gamma_1,\gamma_2)}(x_1, x_2) = f_{\gamma_1}(x_1) f_{\gamma_2}(x_2) = \frac{1}{2\pi} \exp(-\tfrac{1}{2}x_1^2) \exp(-\tfrac{1}{2}x_2^2)$$

and therefore

$$\int_{\{\gamma_1 \in A\}} 1_{\{\gamma_1 \geq \gamma_2\}}\, dP = \frac{1}{2\pi} \int_A \int_{-\infty}^{x_1} \exp(-\tfrac{1}{2}x_1^2) \exp(-\tfrac{1}{2}x_2^2)\, dx_2\, dx_1$$

$$\frac{1}{\sqrt{2\pi}} \int_A \Psi(x_1) \exp(-\tfrac{1}{2}x_1^2)\, dx_1 = \int_{\{\gamma_1 \in A\}} \Psi(\gamma_1)\, d\mathbb{P}.$$

2. a) First we check integrability. We have

$$\mathbb{E}|e^{\eta_n}| = \mathbb{E}e^{\eta_n} = p \sum_{k=1}^{\infty} e^k (1-p)^{k-1} = pk \sum_{j=0}^{\infty} e^j (1-p)^j$$

and this sum is (absolutely) if and only if $1 - p < \frac{1}{e}$, that is, $1 - \frac{1}{e} < p < 1$. Hence, by independence, the ξ_n are integrable (for all $a \in \mathbb{R}$) if and only if $1 - \frac{1}{e} < p < 1$. Next, for these p,

$$\mathbb{E}(\xi_n|\mathcal{F}_{n-1}) = e^{\eta_1+\cdots+\eta_{n-1}-(n-1)a} e^{-a} \mathbb{E}(e^{\eta_n}|\mathcal{F}_{n-1}) = \xi_{n-1} e^{-a} \mathbb{E}e^{\eta_n},$$

since $e^{\eta_1+\cdots+\eta_{n-1}-(n-1)a} = \xi_{n-1}$ is $\mathcal{F}_{n-1}$-measurable and e^{η_n} is independent of $\mathcal{F}_{n-1}$. Thus we find $\mathbb{E}(\xi_n|\mathcal{F}_{n-1}) = \xi_{n-1}$ if and only if $\mathbb{E}e^{\eta_n} = e^a$, that is, if and only if

$$e^a = pk \sum_{j=0}^{\infty} e^j (1-p)^j = \frac{ep}{1 - e(1-p)},$$

that is, $a = \ln \frac{ep}{1-e(1-p)}$.

b) Almost sure convergence is immediate from L^1-boundedness (note that $\xi_n \geq 0$, so $\mathbb{E}|\xi_n| = \mathbb{E}\xi_n = \mathbb{E}\xi_1$) by the martingale convergence theorem.

Pointwise, the convergence implies that either $\eta_1 + \cdots + \eta_n - na \to -\infty$ (in which case the exponential converges to 0) or that $\eta_1 + \cdots + \eta_n - na$ converges to a finite limit. If the latter happens with positive probabiltiy, it would imply that $\eta_n \to a$ with positive probability; this is a contradiction since η_n can take any integer value with finite probability independent of n. It follows that we must have $\eta_1 + \cdots + \eta_n - na \to -\infty$, so $\xi_n \to 0$ almost surely. But $\mathbb{E}\xi_n = \mathbb{E}\xi_1 \neq 0$, so we do not have L^1-convergence.

3. Since M_n is a submartingale, for all N we have

$$\mathbb{E}(1_{\tau\leq N} M_\tau) = \sum_{n=0}^{N} \mathbb{E}(1_{\{\tau=n\}} M_n) \leq \sum_{n=0}^{N} \mathbb{E}(1_{\{\tau=n\}} M_N) = \mathbb{E}M_N,$$

where the used the definition of a submartingale along with the fact that $\{\tau = n\}$ belongs to $\mathcal{F}_n$ in order to get the inequality. By monotone convergence (this uses the nonnegativity), the left-hand side converges to $\mathbb{E}M_\tau$, and the result follows by letting $N \to \infty$.

The game M_n, being a submartingale, is favourable to you. So your win more when you play indefinitely than when you stop.

4. From $\int_0^T B_t dB_t = \frac{1}{2}\left(B_T^2 - T\right)$ we see

$$B_T^2 = T + 2\int_0^T B_t,\, dB_t.$$

Now Itô's formula for $Y_t := tB_t^2$ gives

$$Y_T = TB_T^2 = \int_0^T B_t^2\, dt + 2\int_0^T tB_t\, dB_t + \int_0^T t\, dt$$

and therefore

$$\begin{aligned} G = \int_0^T B_t^2\, dt &= TB_T^2 - \frac{T^2}{2} - 2\int_0^T tB_t\, dB_t \\ &= T\left(T + 2\int_0^T B_t\, dB_t\right) - \frac{T^2}{2} - 2\int_0^T tB_t\, dB_t \\ &= \frac{T^2}{2} + \int_0^T 2\,(T-t)\, B_t\, dB_t \end{aligned}$$

This gives $g(t) = 2\,(T-t)\, B_t$.

5. Note first that $Y_t = y_0 e^{rt}$, so $\tilde{X}_t = y_0^{-1} e^{-rt} X_t$. Since

$$de^{-rt}X_t = -re^{-rt}X_t\, dt + e^{-rt}\, dX_t = e^{-rt}X_t\left[(\mu - r)\, dt + \sigma\, dB_t\right],$$

we may conclude that

$$d\tilde{X}_t = \sigma\tilde{X}_t\left[\frac{\mu-r}{\sigma}\, dt + dB_t\right] = \sigma\tilde{X}_t\, d\tilde{B}_t$$

where $\tilde{B}_t = B_t + \frac{\mu-r}{\sigma}t$. Let $\mathbb{Q}$ be the measure defined by

$$\mathbb{Q}\,(A) = \mathbb{E}_{\mathbb{P}}\left[1_A \exp\left(-\frac{\mu-r}{\sigma}B_T - \left(\frac{\mu-r}{\sigma}\right)^2\frac{T}{2}\right)\right].$$

It follows from Girsanov's theorem that $(\tilde{B}_t)_{t\geq 0}$ is standard Brownian motion, which implies that $(\tilde{X}_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale.