EXAM STOCHASTIC DIFFERENTIAL EQUATIONS (Mastermath)
June 3rd, 2013

1.

(a) State and prove the tower property of the conditional expectation. (6 p)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be convex and increasing. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration. Let $(M_n)_{n\geq 1}$ be a submartingale such that for each $n \geq 1$, $X_n := f(M_n) \in L^1(\Omega)$. Show that $(X_n)_{n\geq 1}$ is a submartingale as well. (6 p)

2. Assume $(X_n)_{n\geq 1}$ is a sequence of independent random variables such that

$$\mathbb{E}(X_n) = \mathbb{E}(X_n^3) = 0, \quad \mathbb{E}(X_n^2) = 1 \quad \mathbb{E}(X_n^4) = \alpha.$$

Let $S_n = \sum_{j=1}^{n} X_j$ and $S_0 = 0$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and define

$$M_n = S_n^4 - 6nS_n^2 + (3-\alpha)n + 3n^2, \quad n \geq 0$$

(a) Show that $(M_{n\geq 0})_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. (7 p)
Hint: Write $S_n = S_{n-1} + X_n$ and use the identity

$$(s+x)^4 = s^4 + 4s^3x + 6s^2x^2 + 4sx^3 + x^4.$$

Next assume $\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = 1/2$ and note that $\alpha = 1$. Let $A \in \mathbb{N} \setminus \{0\}$ and let $\tau = \inf\{n \geq 0 : |S_n| = A\}$. It is known that $\mathbb{E}(\tau) = A^2$ and that τ has finite moments of all orders and use may use both these facts below.

(b) Show that $\mathbb{E}(M_\tau) = 0$. (7 p)
Hint: Use the stopping time theorem and dominated convergence.

(c) Derive that $\mathbb{E}(\tau^2) = \frac{5A^4 - 2A^2}{3}$. (5 p)

3. Assume $(Z_j)_{j\geq 1}$ are independent random variables with normal distribution and $\mathbb{E}(Z_j) = 0$ and $\mathbb{E}(Z_j^2) = 1$. Let $S_n = \sum_{j=1}^n Z_j$ and let $X_n = \exp(S_n - n^\alpha)$, where $\alpha > 0$ is a fixed parameter.

(a) Characterize those $\alpha > 0$ for which one has $\lim_{n\to\infty} X_n = 0$ in L^1. (6 p)
Hint: You may use the identity: $\mathbb{E}(e^{Z_n}) = e^{1/2}$.

(b) Characterize those $\alpha > 0$ for which one has $\lim_{n\to\infty} X_n = 0$ in probability. (6 p)

4. Let $(B_t)_{t\geq 0}$ be a Brownian motion and let $a > 0$.
Define the processes X and Y by $X_t = a^{-1/2}B_{at}$ and $Y_t = B_{2t} - B_t$.

(a) Prove or disprove: $(X_t)_{t\geq 0}$ is a Brownian motion. (5 p)

(b) Prove or disprove: $(Y_t)_{t\geq 0}$ is a Brownian motion. (5 p)

(c) What are the mean and variance of $\int_0^T t^4 B_t \, dB_t$. Explain your answer. (7 p)

5. Let (B_t) be a standard Brownian motion defined on the (filtered) probability space $(\Omega, \mathcal{F}\,(\mathcal{F}_t), P)$. For fixed parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ consider the SDE

$$dX_t = \mu dt + \sigma X_t dB_t, \qquad (*)$$

with the initial condition $X_0 = x_0 \in \mathbb{R}$.

(a) Consider the process $H_t = e^{-\sigma B_t + \frac{1}{2}\sigma^2 t}$. Show that H_t satisfies (7 p)

$$dH_t = -\sigma H_t dB_t + \sigma^2 H_t dt.$$

(b) Suppose X_t is a solution to the SDE $(*)$. Use the (Itô) product rule and (a) to show that (8 p)

$$d(H_t X_t) = \mu H_t dt.$$

(c) Use (b) and the definition of H to show that the solution of $(*)$ is given by (5 p)

$$X_t = x_0 e^{\sigma B_t - \frac{1}{2}\sigma^2 t} + \mu \int_0^t e^{\sigma(B_t - B_s) - \frac{1}{2}\sigma^2 (t-s)} ds.$$

6. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Set $\mathcal{F}_t := \sigma(B_s\,;\, 0 \leq s \leq t)$, $t \geq 0$. Suppose $(X_t)_{0\leq t\leq T}$ satisfies the stochastic differential equation

$$\begin{aligned} dX_t &= rX_t\, dt + \sigma X_t\, dB_t, \quad 0 \leq t \leq T, \\ X_0 &= x_0. \end{aligned}$$

where r, σ and x_0 are positive constants. Using the Girsanov theorem, construct a probability measure under which X_t is an $\mathcal{F}_t$-martingale. (10 p)

$$\text{Grade} = \frac{\text{Number of points}}{10} + 1.$$