$$\frac{6.3}{10}$$

EXAM STOCHASTIC DIFFERENTIAL EQUATIONS (Mastermath)
June 23, 2014, 13:00 – 16:00

1. Let B_t be the standard Brownian motion, $B_T^* = \sup_{0 \le t \le T} B_t$ and $\lambda > 0$.

   (a) Apply Doob's maximal inequality to prove that
   $$P(B_T^* \ge \lambda) \le \frac{T}{\lambda^2}.$$
   [3 pt]

   (b) Sharpen the inequality in (a) by applying the convex function $x \mapsto (x+c)^2$ to B_t for a suitable constant c to prove that
   $$P(B_T^* \ge \lambda) \le \frac{T}{\lambda^2 + T}.$$
   [2 pt]

2. (a) Use the Itô isometry to calculate the mean and the variance of $\int_0^t (B_s + s)dB_s$. [3 pt]

   (b) Calculate the mean and the variance of $\int_0^t (B_s + s)ds$. [4 pt]

3. Let B_t be the standard Brownian motion with respect to the filtration $\mathcal{F}_t$ and let τ_{-1} be the stopping time $\tau_{-1}(\omega) = \inf\{t \colon B_t(\omega) = -1\}$.

   (a) Show that $Z_t = B_{t \wedge \tau_{-1}}$ is a martingale with respect to $\mathcal{F}_t$. (refer to a theorem!) [2 pt]

   We introduce a time-change $X_t(\omega) = Z_{t/(1-t)}(\omega)$ for $0 \le t < 1$. It follows from exercise (a) that X_t is a martingale for $0 \le t < 1$ with respect to the time-change of the filtration.

   (b) Argue that $\lim_{t \to 1} X_t = -1$ with probability one. [1 pt]

   (c) We extend the definition by $X_t = -1$ for $t \ge 1$. Prove that the process X_t is not a martingale. [2 pt]

   (d) Prove that X_t is a local martingale. Use the localizing sequence τ_k that is defined by $\tau_k(\omega) = \inf\{t \colon X_t(\omega) = k\}$ if there exists such a t, or else $\tau_k(\omega) = k$. [2 pt]

4. Use the method of coefficient matching to solve the stochastic differential equation (SDE)
   $$dX_t = -\frac{1}{2} X_t \, dt + \sqrt{1 - X_t^2} \, dB_t \quad \text{where } X_0 = 0.$$
   Look for a solution of the form $u(B_t)$. [3 pt]

   Remark: Note that the diffusion coefficient in the SDE, $\sigma(x) = \sqrt{1 - x^2}$, is not Lipschitz in x near ± 1. So, the existence and uniqueness theorem do not "officially" apply here. This shows that the conditions in the theorem are *not necessary.* They are *sufficient conditions.* Once the "formal" answer by coefficient matching is obtained, you'll see that *if we wanted to* we could prove that our formal answer is an honest answer. We omit this work here, but sometimes it may be necessary.

5. Let $(B_t)_{t \ge 0}$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}(\mathcal{F}_t), P)$. Show that $X_t = e^{\frac{1}{2}t} \cos B_t$ is an $\mathcal{F}_t$-martingale. [3 pt]

   [You may use any theorem, but make sure that the result is applicable by checking all the required conditions.]

6. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}(\mathcal{F}_t), P)$. Suppose $(X_t)_{0\leq t\leq T}$ satisfies the stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= \mu X_t \, dt + \sigma X_t \, dB_t, \quad 0 < t \leq T, \\ X_0 &= x_0, \end{aligned}$$

and $(Y_t)_{0\leq t\leq T}$ evolves deterministically as

$$\begin{aligned} \dot{Y}_t &= rY_t, \quad 0 < t \leq T, \\ Y_0 &= y_0. \end{aligned}$$

where μ, σ, r, x_0 and y_0 are positive constants, and μ is greater than r.

(a) Use Itô formula to find the SDE satisfied by $\tilde{X}_t \equiv \dfrac{X_t}{Y_t}$, $0 \leq t \leq T$. [2 pt]

(b) Using the Girsanov theorem, construct a probability measure under which $\tilde{X}_t$ is an $\mathcal{F}_t$-martingale. [3 pt]

Grade $=$ Number of received points $\times$ 0.3 $+$ 1