Exam Vector Calculus
Module 3 Applied Physics and Applied Mathematics
Bachelor

Codes 202001229-202001232
March 4, 2022, 8.45-10.45

The use of a book is not allowed
All answers must be justified and clearly formulated.

1. Consider a Cartesian coordinate system. Given the domains

   - D_1: triangle with vertices $(0,0), (4,4)$ and $(-4,4)$. The boundary of D_1 is denoted C_1.
   - D_2: circle with radius 1 and center $(0,2)$. The boundary of D_2 is denoted C_2.
   - $D := D_1 \setminus D_2$. Hence D is the domain D_1 with the domain D_2 excluded. The boundary of the domain D is denoted $C = C_1 \cup C_2$.

   The boundary C of the domain D has a positive orientation. The orientation of the boundaries C_1 and C_2 is defined by the orientation of the boundary C.

   Given the vector field

   $$\mathbf{F}(x,y) := F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j} := x^2y\mathbf{i} + y\mathbf{j}.$$

   a. Sketch the domain D. Indicate the proper orientation of the boundary C.

   Note, in the remainder of Question 1 we will only consider the curve C_2.

   b. Give a parametrization of the curve C_2.

   c. Compute $\oint_{C_2} F_1(x,y)dx + F_2(x,y)dy$ without using Green's theorem.

   d. Use Green's theorem to compute $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

   Given the vector field

   $$\mathbf{G}(x,y) := (\sin(y^2) + \exp(x^3))\mathbf{i} + 2xy\cos(y^2)\mathbf{j}.$$

   e. Is the vector field $\mathbf{G}(x,y)$ conservative? Motivate your answer.

   f. Calculate $\oint_{C_2} \mathbf{G} \cdot d\mathbf{r}$. Motivate your answer.

2. The domain $V \subset \mathbb{R}^3$ has the boundary surface $S = S_1 \cup S_2 \cup S_3$ that consists of the parts

$$\begin{aligned} S_1 :=&\{(x,y,z) \in \mathbb{R}^3 \ : \ z = 1 - x^2 - y^2, \ y \geq 0, \ z \geq 0\}, \\ S_2 :=&\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 \leq 1, \ z = 0, \ y \geq 0\}, \\ S_3 :=&\{(x,y,z) \in \mathbb{R}^3 \ : \ z \leq 1 - x^2, \ y = 0, \ z \geq 0\}. \end{aligned}$$

At the boundary surface S the external unit normal vector is given by $\hat{\mathbf{N}}$. Given the vector field

$$\mathbf{F}(x,y,z) := xz\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}.$$

a. Sketch the surface S.

b. Calculate $\oiint_S \mathbf{F} \cdot \hat{\mathbf{N}} dS$ without using Gauss theorem.

c. Calculate $\oiint_S \mathbf{F} \cdot \hat{\mathbf{N}} dS$ using Gauss theorem.

3. Given the oriented surface S, defined as

$$S := \{(x,y,z) \in \mathbb{R}^3 \ : \ z^2 = x^2 + y^2, \ 1 \leq z \leq 2\}.$$

The boundary of the surface S is denoted C and has a positive orientation. The unit normal vector $\hat{\mathbf{N}}$ at S has a negative z-component.

Given the vector field

$$\mathbf{F}(x,y,z) := x^2y\mathbf{i} + xy\mathbf{j} + z^2\mathbf{k}.$$

a. Sketch the surface S. Indicate the orientation of the boundary of S.

b. Calculate **curl F**.

c. Calculate $\iint_S \mathbf{curl\, F} \cdot \hat{\mathbf{N}} dS$ using Stokes' theorem.

Grading

1: 12	2: 15	3: 9
1a: 1	2a: 1	3a: 1
1b: 1	2b: 7	3b: 1
1c: 4	2c: 7	3c: 7
1d: 4		
1e: 1		
1f: 1		

total 36+4=40 points