E&M
UNIVERSITY OF TWENTE.

Boller, Offerhaus, Dhallé

E&M Test 2
31^{st} March 2020

201300164 & 201300183
Fields and electromagnetism

Instructions

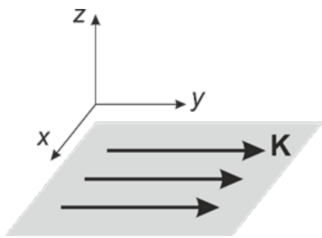
You have **2 hours** to complete the test. Clearly indicate your name and student number on every sheet that you hand in.

You may use the book (Griffiths). No additional notes.

Before answering the questions, read all of them and start with the one you find easiest.

The amount of points to be obtained with each question is indicated next to the question number.

Problem 1 (14 points/100)**.** An infinite flat plate is taken as the *xy*-plane of a Cartesian coordinate system. The plate carries a homogeneous surface current **K** in the positive *y*-direction (Figure on the right).



1.*a.* Copy the figure and sketch a vector representation of the **B**-field at 2 different heights $+z_0$ and $+2z_0$ above the plate, as well as at heights $-z_0$ and $-2z_0$ below the plate.

1.b. Do the same thing for the magnetic vector potential **A**.

Problem 2 (16 points/100)**.** Below you find ten statements. For each of them, indicate whether the statement is 'true' **(T)** or 'not true' **(NT)**. Also include a brief argument why you agree or not (**minimum** 1 & **maximum** 5 lines per statement). Read the statements carefully, each word may be important!

2.*a.* Inside an infinitely long and current-carrying solenoid, a positive charge q is moving parallel to the axis of the solenoid. Such a charge will experience a Lorentz force that points radially inwards towards the axis of the solenoid.

2.*b.* Magnetic field lines always form closed loops, but they can cross each other.

2.*c.* In a part of space where the volumetric current density is uniform, the magnetic field has to be constant.

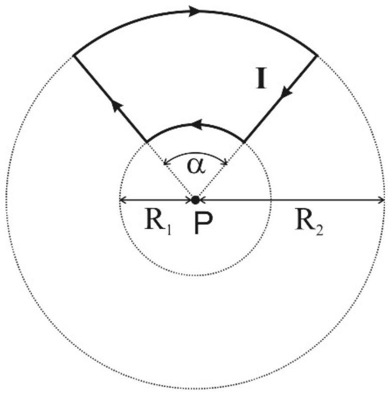
2.*d.* When the current in a coil is increased from I_0 to $2I_0$, the magnetic energy stored in the system also doubles.

2.*e.* The total magnetic **B**-flux through a closed surface is always zero, irrespective of possible material interfaces or free currents that cut the surface.

2.*f.* In a static situation, the divergence of the volumetric current density is always zero.

2.*g.* The space in-between 2 concentric spherical metal shells is filled with a uniform material (with electrical conductivity s) and a constant current flows from the inner shell to the outer one. The electric field in-between the shells decreases with increasing radial distance r to the centre as $E(r) \sim 1/r$.

2.*h.* When one moves a wire-frame through a uniform magnetic field, no emf is induced as long as the orientation of the frame with respect to the field does not change.



Problem 3 (20 points/100)**.** A conducting wire frame in the *xy*-plane (the thick solid line in the figure) carries a current ***I***. The frame consists of two straight sections at an angle α and two concentric arcs with radii R_1 and R_2.

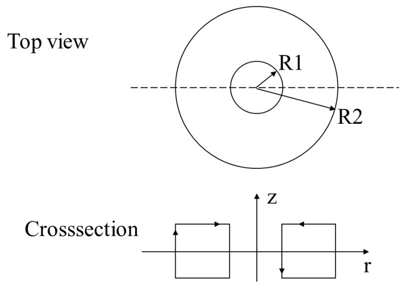
We wish to calculate the magnetic induction ***B***(P) in the centre point P (the crossing of the two straight sections) using the Biot-Savart law:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int \frac{\mathbf{d}\boldsymbol{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

3.*a.* Show that $\mathbf{B}(\mathrm{P}) = \frac{\mu_0 I}{4\pi}\frac{R_2 - R_1}{R_1 R_2}\alpha\,\hat{\mathbf{z}}$ (*hint: use cylindrical coordinates*).

3.*b.* Use the expression derived in 3.*a* to calculate the magnetic induction $\boldsymbol{B}_{\text{loop}}$ generated in the centre of a single current-carrying circular loop and explain how you get this from 3.*a*.

Problem 4 (25 points/100)**.** The figure to the right shows a torroidial coil with a square crossection that has N turns and carries a current I. The inner radius is R_1, the outer radius is R_2.



4.*a.* Calculate the magnetic induction ***B*** as a function of *r* (between R_1 and R_2). The number of turns N is high enough for the coil to be wound tightly so that stray fields may be ignored..

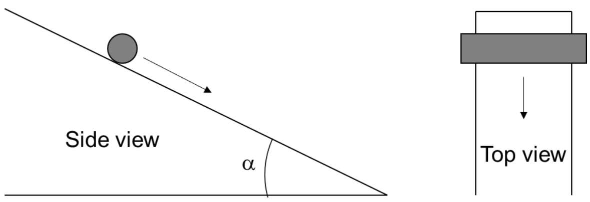
Then the coil is filled with a material with a μ_r of 50. The magnetisation causes bound surface- and volume- current to flow.

4.*b.* Calculate the new ***B*** field and the magnetisation ***M***.

4.*c.* Calculate the magnitude of the bound surface- and volumetric current densities K_B and J_B as a function of *r*.

4.*d.* Copy the cross-section and sketch the direction of all bound currents.

Problem 5 (25 points/100)**.** Two parallel perfect conductors (no resistance) are separated by a distance L and run down at an angle α compared to the horizontal. A cylinder of mass M and total electrical resistance R is rolling down over the conductors. The conductors are also connected at the top. In the space in-between the conducting rails is a vertical homogeneous magnetic field B_o. Suppose that the rolling cylinder has an initial speed **v** (pointing along the rails).



5.*a.* Show that for a given speed *v*, the induced current through the resistor is given by

$$I_{ind} = \frac{B_0 L v \cos\alpha}{R}$$

5.*b.* Calculate the Lorentz force on the cylinder for a given speed **v**. What is the direction of this Lorentz force?

5.*c.* The cylinder is accelerated downwards by the component of gravity along the track (Mg sinα). What will be the final speed of the conductor?

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

ϵ_0	$= 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$= 4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$= 3.00 \times 10^{8}\,\mathrm{m/s}$	(speed of light)
e	$= 1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$= 9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Standaardintegralen .

$I = \int x^m (a^2+x^2)^n\,dx$ Noem $Y = \sqrt{a^2+x^2}$; $Y^2 = a^2+x^2$;

m	n	I	m	n	I
-2	-1/2	$-Y/(a^2x)$	1	-3/2	$-1/Y$
-2	-1	$-a^{-2}(\frac{1}{x}+\frac{1}{a}\arctan\frac{x}{a})$	1	-1	$\ln\lvert Y\rvert$
-1	-3/2	$a^{-2}(\frac{1}{Y}-\frac{1}{a}\ln\left\lvert\frac{a+Y}{x}\right\rvert)$	1	-1/2	Y
-1	-1/2	$-(1/a)\ln\lvert(a+Y)/x\rvert$	1	1/2	$\frac{1}{3}Y^3$
-1	-1	$a^{-2}\ln\lvert x/Y\rvert$	1	3/2	$\frac{1}{5}Y^5$
0	-3/2	$x/(a^2Y)$	2	-3/2	$\ln\lvert x+Y\rvert - x/Y$
0	-1	$a^{-1}\arctan(x/a)$	2	-1	$x - a\arctan(x/a)$
0	-1/2	$\ln\lvert x+Y\rvert$	2	-1/2	$\frac{1}{2}xY - \frac{1}{2}a^2\ln\lvert x+Y\rvert$
0	1/2	$\frac{1}{2}xY + \frac{1}{2}a^2\ln\lvert x+Y\rvert$	2	1/2	$\frac{1}{8}x(2x^2+a^2)Y - \frac{1}{8}a^4\ln\lvert x+Y\rvert$
0	3/2	$\frac{1}{8}x(2x^2+5a^2)Y + \frac{3}{8}a^4\ln\lvert x+Y\rvert$	3	-3/2	$Y + a^2/Y$
			3	-1/2	$\frac{1}{3}Y^3 - a^2Y$
			3	1/2	$\frac{1}{5}Y^5 - \frac{1}{3}a^2Y^3$

$I = \int \sin^m ax \cos^n ax\,dx$

m	n	I	m	n	I
1	0	$-(1/a)\cos ax$	1	1	$(\sin^2 ax)/2a$ of $-(\cos^2 ax)/2a$
0	1	$(1/a)\sin ax$	2	2	$-\frac{1}{32a}\sin 4ax + \frac{x}{8}$
1	-1	$-(1/a)\ln\lvert\cos ax\rvert$	1	n	$-\frac{\cos^{n+1} ax}{(n+1)a}$
-1	1	$(1/a)\ln\lvert\sin ax\rvert$	m	1	$\frac{\sin^{m+1} ax}{(m+1)a}$
2	0	$\frac{1}{2}x - \frac{1}{4a}\sin 2ax$	0	2	$\frac{1}{2}x + \frac{1}{4a}\sin 2ax$
3	0	$-\frac{1}{3a}\cos ax(\sin^2 ax + 2)$	0	3	$\frac{1}{3a}\sin ax(\cos^2 ax + 2)$
4	0	$\frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$	0	4	$\frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

6.5 Benaderingen voor $|x| \to 0$

$(1+x)^a$	$1+ax+\ldots$	$\sin x$	$x - x^3/6+\ldots$
e^x	$1+x+\ldots$	$\cos x$	$1 - x^2/2+\ldots$
$\ln(1+x)$	$x - x^3/2+\ldots$		