Test 2 for Probability Theory
(Module Signals and Uncertainty, 201800138)
Monday June 15, 2020, 8.45 - 11.15 hours.

This test consists of 4 problems and 1 table (P.T.O.)
Use proper notation and motivate all answers.

Integrity statement

Please read the following paragraph carefully, copy the text below it verbatim to the first page of your work (handwritten) and sign it with your signature. If you fail to do so, your test will not be graded.
By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.
The only allowed sources for this test are:

- the book Introduction to Probability Models by Ross (hardcopy or pdf)
- the slides (printed or pdf)
- electronic devices, but only to be used:
  - for downloading the test and afterwards uploading your work to Canvas
  - to show the test/book/slides on your screen
  - to write the test (in case you prefer to use a tablet instead of paper to write on)

Text to be copied (handwritten) and signed:

I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.

Norm:

1					2			3					4			Total
a	b	c	d	e	a	b	c	a	b	c	d	e	a	b	c	
2	3	2	2	3	2	3	2	2	2	2	3	1	2	3	2	36

Grade: $\dfrac{\text{Total}}{4} + 1$

Note: the grade of this second test is 70% of the final grade for Probability Theory (first test is 30%).

1. Let $X_i, i = 1, 2, \ldots$ be independent and exponentially distributed random variables, each with expectation λ^{-1}. Let N be a geometric random variable with parameter p, independent of the X_i, and let $S_n = \sum_{i=1}^{n} X_i$.

   a. Determine $E[S_N|N]$.

   b. Determine $\text{Var}[S_N]$.

   c. Determine $\text{Cov}(S_2, X_1)$.

   d. Give the moment generating function of $S_n, n = 1, 2, \ldots$.

   e. Use d. above to show that S_N has an exponential distribution.

2. In a factory, a scale is used to determine the weights of boxes, filled with various types of products. The scale will break if the total weight exceeds 144 kg. The weights of n boxes are modeled as (independent) random variables $X_1, \ldots, X_n$. Suppose all X_i have a normal distribution with expectation 15 kg and variance 4 kg^2.
   Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n}$ be the average weight of the n boxes.

   a. Determine the probability distribution of $\bar{X}$.

   b. Determine the probability the scale will break if we load it with $n = 9$ boxes.

   Now assume the weight of each box is uniformly distributed on $[a, b]$, with the same expectation 15 kg and variance 4 kg^2.

   c. Determine a and b, and approximate the probability the scale will break if we load it with 9 boxes.

3. The random variables X and Y have joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{1-y} & \text{if } \ x > 0, \ y > 0 \text{ and } x + y < 1, \\ 0 & \text{else.} \end{cases}$$

   a. Show that f is indeed a probability density function.

   b. Determine the marginal densities of X and Y.

   c. Determine $f_{X|Y}(x|y)$, the conditional density of X, given $Y = y$ for $0 < y < 1$.

   d. Determine $P(X < Y)$.

   e. Are X and Y independent?

4. The random variables X and Y are independent and exponentially distributed with expectation $1/3$.

   a. Determine $P(X > 3 \,|\, X > 1)$.

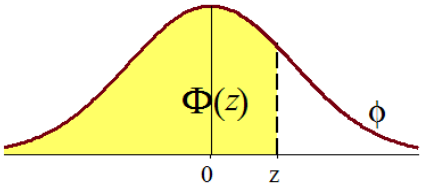
   b. Determine $E[X \,|\, Y > 1]$ and $E[Y \,|\, Y > 1]$.

   c. Determine $E[\min(X, Y)]$ and $E[\max(X, Y)]$.

Standard normal probabilities

The table gives the distribution function Φ for a N(0,1)-variable Z

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx$$



Last column: N(0,1)-density function (z in 1 dec.): $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

	Second decimal of **z**										
z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**φ(z)**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	0.3989
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	0.3970
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	0.3910
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	0.3814
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	0.3683
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	0.3521
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	0.3332
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	0.3123
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	0.2897
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	0.2661
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	0.2420
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	0.2179
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	0.1942
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	0.1714
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	0.1497
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	0.1295
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	0.1109
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	0.0940
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	0.0790
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	0.0656
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0.0540
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0.0440
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0.0355
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0.0283
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0.0224
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0.0175
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0.0136
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0.0104
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0.0079
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0.0060
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990	0.0044
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	0.0033
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0.0024
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0.0017
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.0012
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.0009
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.0006