Make-up Exam: Analysis II

Statistics and Analysis (202001350)

November, 11, 2021, 08:45 – 11:45 Total Points : 35

All answers must be motivated.
Approach to a solution is equally important as the final answer.
The exam is closed-book and also, use of an electronic calculator is not allowed
Good Luck!

1. Let the sequence $\{a_k,\ k \in \mathbb{N}\}$ be such that:

$$\text{The infinite series } \sum_{k=1}^{\infty} a_k \text{ converges.} \tag{1}$$

Indicate which of the following statements are true and which are false. If the statement holds, then provide a proof, when false provide a counter example.

(a) For all sequences $\{a_k,\ k \in \mathbb{N}\}$ satisfying (1) we have; [2]

$$\sum_{k=1}^{\infty} |a_k| < \infty.$$

(b) For all sequences $\{a_k,\ k \in \mathbb{N}\}$ satisfying (1) we have; [2]

$$\sum_{k=1}^{\infty} a_k^2 < \infty.$$

(c) For all sequences $\{a_k,\ k \in \mathbb{N}\}$ satisfying (1) we have; [2]

$$\text{The infinite series } \sum_{k=1}^{\infty} \frac{a_k}{k} \text{ converges.}$$

2. Find the radius of convergence and the convergence interval of the power series: [4]

$$\sum_{k=1}^{\infty} \frac{3^k}{\sqrt{k}}(x-4)^k.$$

3. (a) Give the definition of uniform convergence of a sequence of real-valued functions, using ϵ-δ-N arguments/language. [1]

Define the following sequence of functions $f_n : \mathbb{R} \mapsto \mathbb{R}$

$$f_n(x) = \sin(x/n), \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}$$

(b) Show that f_n converges uniformly on every bounded interval $(a, b) \subset \mathbb{R}$. What is its limit function? [3]

(c) Show that f_n does not converge uniformly on $\mathbb{R}$. [1]

4. Prove that the following function is analytic on $(-1,1)$ and determine its Maclaurin expansion. [3]

$$h(x) = \frac{1}{(x+1)^2}.$$

5. Consider on $\mathbb{Z}$ the following metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2} & \text{if } |x-y| = 1 \\ 2 & \text{if } |x-y| \geq 2. \end{cases}$$

Show that d does not define a metric on $\mathbb{Z}$. [2]

6. Let (X,ρ) and (Y,τ) be metric spaces.

(a) Give the definition of a compact set in X. [1]

(b) Let f be a continuous function from the metric space (X,ρ) to the metric space (Y,τ), and let C be a compact set in X. Prove that $f(C)$ is a compact set in Y. [2]

(c) Let f be a continuous function from the metric space (X,ρ) to the metric space (Y,τ), and let $f(V)$ be a compact set in Y. Is V compact in X? If this statement holds, then provide a proof, when false provide a counter example. [2]

7. Consider the function $q : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x,y) = \begin{cases} |xy| \log(x^2+y^2) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) At which points of $\mathbb{R}^2 \setminus \{0,0\}$ is f differentiable? [2]

(b) Prove that f is differentiable at $(0,0)$. [2]

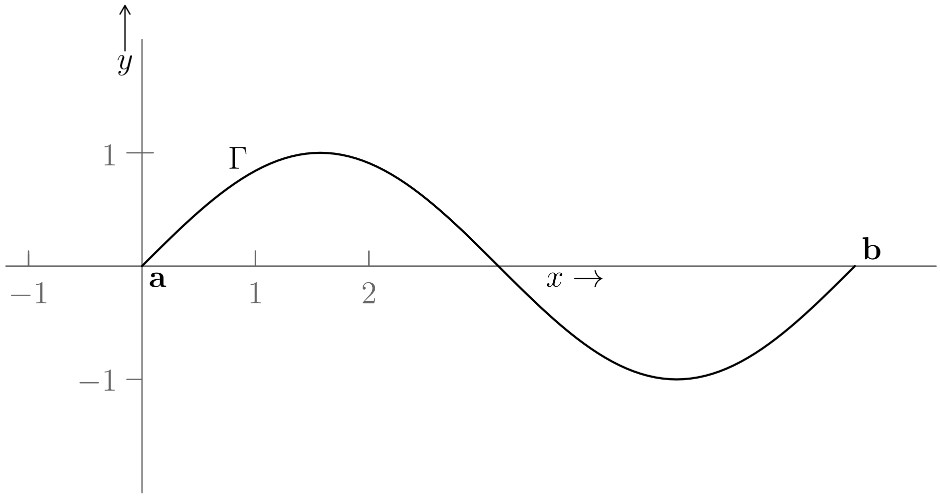

Figure 1: The sin-curve Γ in $\mathbb{R}^2$

8. In $\mathbb{R}^2$ we consider the curve $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 2\pi], y = \sin(x)\}$, as drawn in Figure 1. We take the points $\mathbf{b} = (2\pi, 0)$ and $\mathbf{a} = (0, 0)$.

   Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be continuous differentiable. Prove that there exists a point $\mathbf{c} = (c_1, c_2)$ on Γ such that [3]

$$f(\mathbf{b}) - f(\mathbf{a}) = 2\pi \nabla f(\mathbf{c}) \cdot \begin{bmatrix} 1 \\ \cos(c_1) \end{bmatrix}.$$

9. Prove that there exist functions $u(x, y), v(x, y)$, and $w(x, y)$, and an $r > 0$ such that u, v, w are continuous differentiable and satisfy the equations

$$\begin{aligned} u^3 + xv^3 + xy - w &= 0, \\ v^2 - yw^2 + x &= -2, \\ uvw - xy &= -1, \end{aligned}$$

   on $B_r(1, 1)$ and satisfy $u(1, 1) = 0$, $v(1, 1) = 1$, $w(1, 1) = 2$. [3]

Grade:	$\dfrac{\text{score on test}}{35} \times 9 + 1$		(rounded off to one decimal place)