1. (a) $\mathscr{R} = \begin{bmatrix} 1 & 6 \\ 1 & \alpha+1 \end{bmatrix}$❶. So controllable iff $\alpha \neq 5$❶

   (b) *Method 1:* $\begin{bmatrix} sI-A \\ C \end{bmatrix} = \begin{bmatrix} s-2 & -4 \\ -\alpha & s-1 \\ 1 & 1 \end{bmatrix}$ loses rank iff the columns are the same: $s=-2$ and $\alpha=3$. So it does not lose rank for $\Re(s) \geq 0$. So detectable.❸

   Method 2: $\mathscr{W} = \begin{bmatrix} 1 & 1 \\ 2+\alpha & 5 \end{bmatrix}$ so not observable iff $\alpha = 3$. Hence for sure it is detectable for $\alpha \neq 3$❶. For $\alpha = 3$ the eigenvalues of A are 5 and -2. At the unstable eigenvalue $s=5$ the "Hautustest" gives $\begin{bmatrix} sI-A \\ C \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -3 & 4 \\ 1 & 1 \end{bmatrix}$❶ which has full column rank❶. So also for $\alpha = 3$ it is detectable. So always detectable.

   (c) $F = \begin{bmatrix} -4 & -4 \end{bmatrix}$❷. (Fun: the answer doesn't depend on α)

   (d) Can choose eigenvalues -1, which gives $L = \begin{bmatrix} 3\frac{2}{3} \\ 1\frac{1}{3} \end{bmatrix}$. You may also guess something as long as $A-LC$ is as.stable, e.g. $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ for then $A-LC = \begin{bmatrix} 0 & -2 \\ -2 & -1 \end{bmatrix}$ which is as.stable. I'll choose the latter: $L = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (correct L❶). Then the controller is

   $$\dot{\hat{x}} = (A - LC + BF)\hat{x} + Ly, \qquad u = F\hat{x}❶$$

   which for my L is

   $$\dot{\hat{x}} = \begin{bmatrix} -4 & -2 \\ -6 & -5 \end{bmatrix}\hat{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} y, \qquad u = -4\hat{x}.$$

2. $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + 2u❸$$

3. (a) $-PK/(1+PK)$❶ (correct derivation: another❶)

   (b) $K(s) = s+1$ would do for then $\chi_{\text{closed}} = s^2 + s + 1$ which is as.stable.❸ (easy points)

   (c) $\lim_{t\to\infty} y(t) = H_{y/m}(0) m_0 = -m_0$..❷.

4. (a) $\dot{y} = -y + w$ (with input w) is BIBO because it is asymptotically stable❶. $w(t) := \int_{t-2}^{t} u(\tau)\,\mathrm{d}\tau$ (with input u and output w) is BIBO because $|w(t)| \leq \int_{t-2}^{t} \|u\|_\infty \,\mathrm{d}\tau = 2\|u\|_\infty$. So $\|y\|_\infty \leq 2M\|u\|_\infty$ where M is maximal peak-gain of $y = -\dot{y} + w$. So BIBO. ❶

   (b) Plug in $u(t) = \mathrm{e}^{\mathrm{i}\omega t}$ and $y(t) = H(\mathrm{i}\omega)\,\mathrm{e}^{\mathrm{i}\omega t}$[half point] gives

   $$H_{y/u}(\mathrm{i}\omega) = \frac{1 - \mathrm{e}^{-2\mathrm{i}\omega}}{\mathrm{i}\omega(\mathrm{i}\omega + 1)}❷$$

   (for completeness: at $\omega = 0$ we have $H_{y/u}(0) = 2$ so DC-gain is 2.)

   (c) The impulse response of $\dot{y} = -y + w$ is $h_{y/w}(t) = \mathrm{e}^{-t}\mathbb{1}(t)$ so its maximal peak-to-peak gain is $M := \int_0^\infty h(t)\,\mathrm{d}t = 1$. Hence the maximal peak-to-peak gain of $\dot{y} = -y + \int_{t-2}^{t} u(\tau)\,\mathrm{d}\tau$ is at most $2M = 2$. This equals the DC-gain, so the maximal possible peak-to-peak gain of

2 is attained for constant inputs $u(t) = u_0$ (hence also for $u(t) = u_0 \mathbb{1}(t)$). ❶

Method 2: Compute impulse response from $\dot{h}(t) = -h(t) + \mathbb{1}(t) - \mathbb{1}(t-2)$. So $h(t) = 1 - \mathrm{e}^{-t}$ on $[0,2]$ and $h(t) = (\mathrm{e}^2 - 1)\mathrm{e}^{-t}$ for $t > 2$. Then compute $\int |h| = \int h = 2$.

Method-3: argue that $h(t) \geq 0$ then use an exercise in the notes that claims that then $|H(0)| = 2$ is its maximal peak-to-peak gain.

5. (a) see lecture notes.. ❷

   (b) *Method 1:* No. The response to $u_0(t) = \mathbb{1}(t)$ is $y_0(t) = \mathbb{1}(-t)$ while the response of the delay $u_1(t) = u_0(t-1) = \mathbb{1}(t-1)$ is $y_1(t) = \mathbb{1}(-t-1)$ which is not the same the delay of the response $y_0(t-1) = \mathbb{1}(-(t-1))$. ❷

   Method 2: Its impulse response is $h(t) = \delta(-t) = \delta(t)$. Then LTI would mean that $y(t) = (h * u)(t) = \int_\tau \delta(t-\tau)u(\tau)\mathrm{d}\tau = u(t)$ which it isn't so not LTI.

   (c) If $u = 0$ then $x(t) = \mathrm{e}^t x_0$ and $y = \mathrm{e}^{2t} x_0^2$ from which it is impossible to determine the sign of x_0: not observable. ❷

6. (a) Since $\lim_{x \downarrow -\pi/2+k\pi} \tan(x) - x = -\infty$ and $\lim_{x \downarrow +\pi/2+k\pi} \tan(x) - x = \infty$ it has at least one zero on $]-\pi/2+k\pi, \pi/2+k\pi[$ because of continuity❶. There is *precisely one* zero because the derivative of $\tan(x) - x = \frac{1}{\cos^2(x)} - 1 = \tan^2(x)$ is > 0 almost everywhere (so $\tan(x) - x$ strictly increasing)❶.

   (b)

      i. bisection❶

      ii. Initial $x_L = \pi - \pi/2$ and $x_R = \pi + \pi/2$ work. Each bisection halves the length of the interval, so about 15 steps are needed because $\pi 2^{-15} = 9.6 \times 10^{-5} \approx 10^{-4}$❶

   (c) i. $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{\tan(x_0) - x_0}{1/\cos^2(x_0) - 1} = \infty$ because the derivative is zero..❶.

      ii. ❶ Assuming the method would converge then (assuming $f'(x) \neq 0$ around the zero) the error roughly quadruples every step. If the initial error would have been about 10^{-1} then in the next step about 10^{-2} and then 10^{-4}. So then three steps would have been sufficient.... (a bit vague).