MSc course 191551150 "Numerical techniques for PDEs"
Practice final test, January 8, 2015

The parts A and B may be graded separately. Therefore please write your answers for the A and B parts on separate sheets of paper. The use of calculators and other electronic devices is not allowed. Motivate all your answers.

Part A. Parabolic PDEs Consider the following PDE

$$u_t - (Du_x)_x = 0, \tag{1}$$

with initial and boundary conditions given, where $u = u(x,t)$ is unknown and $D = D(x) > 0$ is given.

1pt **A1** Give a definition of a convergent numerical scheme for solving equation (1).

2pt **A2** Assume that for the truncation error $T(x,t)$ of some numerical scheme applied to solve (1) holds

$$|T(x,t)| \leqslant C\Delta t,$$

where C is a constant (possibly depending on the CFL number). First, define a maximum error at a time step, denoted by E^n, of a numerical scheme for solving equation (1). After that, show that $E^n \to 0$ as $\Delta t \to 0$ while the CFL number is kept constant.

3pt **A3** Consider the following scheme to solve (1):

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{D_{j+1/2}(U_{j+1}^n - U_j^n) - D_{j-1/2}(U_j^n - U_{j-1}^n)}{(\Delta x)^2} = 0,$$

where $D_{j\pm 1/2} = D(x_{j\pm 1/2})$. Carry out the practical check for the maximum principle to see whether the principle holds for some Δt. Hint: (i) check whether the update coefficients are positive for some CFL number ν; (ii) check the sum property.

Part B. Hyperbolic PDEs Consider the following hyperbolic PDE with some initial and boundary conditions given:

$$u_t + au_x = 0, \tag{2}$$

where $u = u(x,t)$ is unknown and $a = a(x,t)$ is given.

1pt **B1** Provide a definition of the CFL condition for equation (2).

3pt **B2** Consider the following numerical scheme for solving PDE (2):

$$\frac{U_j^{n+1} - U_j^n + U_{j+1}^{n+1} - U_{j+1}^n}{2\Delta t} + a\frac{U_{j+1}^n - U_j^n + U_{j+1}^{n+1} - U_j^{n+1}}{2\Delta x} = 0, \tag{3}$$

where $a = a(x_{j+1/2}, t_{n+1/2})$. First, give a definition of the truncation error of a numerical scheme. After that investigate the order of the truncation error of the scheme in space and time. Hint: for all the terms in the scheme, apply the Taylor expansion around the point $(x_{j+1/2}, t_{n+1/2})$.

2pt **B3** We analyze the damping and phase error of a numerical scheme (which is not necessarily the scheme given in (3)) for solving hyperbolic PDE (2). To do so, we substitute a numerical Fourier mode $U_j^n = \lambda^n e^{ikj\Delta x}$ into the scheme and obtain

$$\lambda = 1 - \nu + \nu e^{-i\xi}, \qquad \nu = a\frac{\Delta t}{\Delta x}, \quad \xi = k\Delta x.$$

First, describe, without carrying out the detailed analysis, how, based on the provided data, you would judge the damping and phase error of the scheme. After that, carry out the analysis for the damping error, to obtain an expression of the form

$$\text{damping error} = O(\xi^p),$$

where p is an integer.

The grade for the test is determined as $G = 1 + 9P/12$ where P is the number of points earned.