# MSc course 191551150 "Numerical techniques for PDEs"
# Final test, January 14, 2016, 08:45–10:45

The parts A and B may be graded separately. Therefore please write your answers for the A and B parts on separate sheets of paper, indicating your name on both A and B parts. The use of calculators and other electronic devices is not allowed. Motivate all your answers.

**Part A. Parabolic PDEs**

The challenge is to determine the evolution of a concentration field $u$ due to the action of nonlinear diffusion. In one spatial dimension this is governed by

$$u_t - (D(u)u_x)_x = 0 \quad ; \quad t > 0,\ 0 < x < 1$$

The diffusion process depends nonlinearly on the solution $u$. We assume that $D > 0$ and $0 \leq u \leq 1$ for all $x$ and $t$. Here, we assume that

$$D(u) = (1 - au)^4$$

where $0 \leq a < 1$. As initial and boundary conditions we adopt

$$u(x, 0) = x \quad ; \quad u_x(0, t) = u_x(1, t) = 0$$

i.e., a linear concentration profile and homogeneous Neumann conditions. We set ourselves the goal to develop an explicit scheme for nonlinear diffusion.

1pt **A1** Discretize the nonlinear diffusion equation on a uniform grid $x_j = j\Delta x$; $j = 0, 1, \ldots, J$, using the explicit scheme, central differencing and an implementation of the boundary conditions based on the ghost-cell method.

2pt **A2** Determine the order of accuracy of the scheme.

3pt **A3** Assume $m \leq D(u) \leq M$, $m > 0$. Propose a limitation for the time-step $\Delta t$ that yields a stable time-integration.

**Part B. Hyperbolic PDEs**

1pt **B1** To solve a hyperbolic PDE

$$u_t - au_x = 0,$$

with $u = u(x, t)$ unknown and $a = \text{const} < 0$ given, we apply a numerical scheme

$$U_j^{n+1} = U_j^n - a\Delta t \frac{U_{j+1}^n - U_j^n}{\Delta x}.$$

It is known that this scheme is stable provided that $\frac{\Delta t|a|}{\Delta x} \leqslant 1$. Does this scheme satisfy the CFL restriction for $\frac{\Delta t|a|}{\Delta x} \leqslant 1$?

See the other side

2pt **B2** Consider a hyperbolic PDE

$$u_t - a u_x = 0,$$

with $u = u(x,t)$ unknown and $a = \text{const} > 0$ given. We carry out a Fourier analysis for a certain numerical scheme to find expressions for its damping and phase errors. Assume that the exact and numerical Fourier modes are respectively given by the following expressions:

$$u_{\text{exact}} = \mathrm{e}^{\mathrm{i}(kx+\omega t)}, \qquad U_j^n = \lambda^n \mathrm{e}^{\mathrm{i}kj\Delta x},$$

where the notation is as usual. Give the (general) definition of the phase error of a numerical scheme.

3pt **B3** Consider a vector hyperbolic PDE

$$\boldsymbol{u}_t - A\boldsymbol{u}_x = 0,$$

where $A$ is a constant $n \times n$ matrix and $\boldsymbol{u}(x,t)$ is an unknown vector function whose values are vectors in $\mathbb{R}^n$. To solve this equation, we apply the following numerical scheme:

$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n - \frac{\Delta t}{\Delta x} A \Delta_{0x} \boldsymbol{U}_j^n + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2 \delta_x^2 \boldsymbol{U}_j^n,$$

where the familiar notation is used:

$$\Delta_{0x}\boldsymbol{U}_j^n = \frac{1}{2}(\boldsymbol{U}_{j+1}^n - \boldsymbol{U}_{j-1}^n), \qquad \delta_x^2 \boldsymbol{U}_j^n = \boldsymbol{U}_{j+1}^n - 2\boldsymbol{U}_j^n + \boldsymbol{U}_{j-1}^n.$$

Assume that a numerical Fourier mode has a form

$$\boldsymbol{U}_j^n = \lambda^n \mathrm{e}^{\mathrm{i}kj\Delta x}\boldsymbol{U},$$

where $\boldsymbol{U} \in \mathbb{R}^n$ is a constant nonzero vector and the other notation is as usual. Carry out a Fourier analysis of the scheme with the given Fourier mode to show that

$$\lambda \boldsymbol{U} = B\boldsymbol{U}, \qquad B \in \mathbb{R}^{n \times n},$$

and provide an expression for the matrix $B$ in terms of $A$, $\xi = k\Delta x$, $\Delta t$, $\Delta x$ and i.

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The grade for the test is determined as $G = 1 + 9P/12$ where $P$ is the number of points earned.